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A HOMOGENIZATION PROCEDURE AND A PHYSICAL DISCRETE MODEL FOR GEOMETRICALLY NONLINEAR TRANSVERSE VIBRATIONS OF A CLAMPED BEAM MADE OF A FUNCTIONALLY GRADED MATERIAL

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Abstract

Functionally graded materials are used in aircrafts, space vehicles and defence industries because of their good thermal resistance. Geometrically nonlinear free vibration of a functionally graded beam with clamped ends (FGCB) is modeled here by an *N*-dof discrete system presenting an equivalent isotropic beam, with effective bending and axial stiffness parameters obtained via a homogenization procedure. The discrete model is made of *N* masses placed at the ends of solid bars connected by rotational springs, presenting the flexural rigidity. Transverse displacements of the bar ends induce a variation in their lengths giving rise to axial forces modeled by longitudinal springs. The nonlinear semi-analytical model previously developed is used to reduce the vibration problem, via application of Hamilton's principle and spectral analysis, to a nonlinear algebraic system involving the mass and rigidity tensors m_{ij} and k_{ij} and the nonlinearity tensor b_{ijkl}. The material properties of the (FGCB) examined is assumed to be graded according to a power rule of mixture in the thickness direction. The fundamental nonlinear frequency parameters found for the (FGCB) are in a good agreement with previously published results showing the validity of the present equivalent discrete model and its availability for further applications to non-uniform beam.

Keywords: Discrete system, Clamped beam, Nonlinear transverse vibrations, FGMs, Homogenization procedure.

1. INTRODUCTION

The functionally graded materials (FGM) are a type of inhomogeneous materials made of a mixture of ceramic and metal in proportions varying from 0 to 1 in the structure thickness direction, as shown in Figure 1 [1 to 9]. These materials have been designed by researchers in Japan (1980) in order to overcome some of the disadvantages encountered with classical composites such as the stress concentration at the interfaces between the layers due to the sharp change in mechanical properties from one layer to another. Nowadays, FGM's are being increasingly used in many fields, such as aircrafts, space vehicles and defence industries, electronic and biomedical equipments. Due to their good thermal resistance, they are also used in areas such as nuclear engineering, aeronautics, etc.

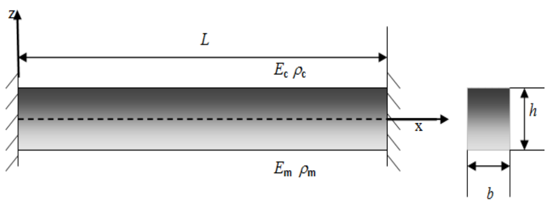

Fig. 1. FG beam notation

Geometrical nonlinearity occurs in vibrating structures when the amplitude of vibration is so large that it cannot be considered as small compared to the structure thickness. This may be due to the proximity of one of the resonance frequencies or to high excitation levels due to severe environments. One of the main effects of non-linearity is the dependence of the natural frequencies on the amplitude of vibration. To make a safe and an efficient design in such situations, non-linear models should be available to analyse properly the system dynamics and estimate accurately the dynamic characteristics amplitude dependence.

The present work is concerned with geometrically non-linear vibrations of a functionally graded beam with clamped ends (FGCB). The FG beam is modelled by a discrete *N*-dof system presenting the equivalent isotropic beam obtained via a homogenization procedure. In [10 and 11], a discrete model, shown in Figure 2, has been developed allowing the study of nonlinear geometrical bending vibrations of homogeneous beams. Euler-Bernoulli's first-order theory of neglecting the transverse shear effect for long beams has been applied. Various applications have followed to nonlinear vibrations of non-uniform beams, beams carrying point masses, beams resting on elastic foundations and cracked beams [12 to 14].

The discrete model mentioned above is made of *N* point masses and (*N*+2) torsional springs with a rigidity C_r. The spiral springs are attached to (*N*+1) massless bars considered as longitudinal springs of length *l* and stiffness *k* in order to take into account the axial forces induced by the geometrical non-linearity. Indeed, severe work conditions and repeated loading cause a material fatigue which may be accelerated when resonance occurs around the structure natural frequencies, inducing large vibration amplitudes. If such a nonlinear effect is not well controlled, it may lead to structural damages and catastrophic failures.

. Beam with clamped ends (FGCB) is modelled here by the *N*-dof discrete system presenting an equivalent isotropic beam, with effective bending and axial stiffness parameters obtained via a homogenization procedure see Figure 2.

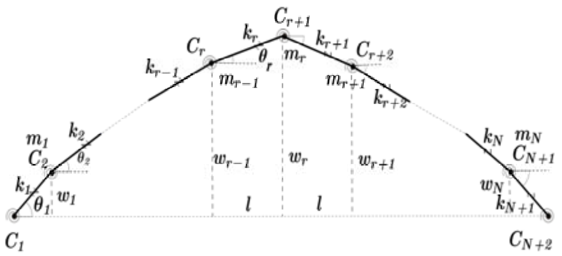

Fig. 2. FGCB modelled by multi-degree-of-freedom discrete system with effective bending and axial stiffness parameters obtained via a homogenization procedure.

The nonlinear vibration analysis becomes then essential for a reliable structural design. The transverse variation of the (FGCB) dynamic characteristics through the beam thickness presented in [9] is adopted in this work, the axial inertia and damping are ignored and a homogenization procedure is proposed to reduce the problem studied to that of an equivalent homogeneous beam with effective bending and axial stiffness parameters [9]. The analogy between the classical continuous model of the (FGCB) and the present discrete model is developed. Then, the fundamental amplitude dependent nonlinear frequency parameters found here for the (FGCB) are compared to previously published results in order to validate the equivalent discreet model and to show its availability for further applications to non uniform FG beam with any type of ceramic distribution.

2. THEORETICAL FORMULATION FOR A CLAMPED FUNCTIONALLY GRADED BEAM

In this section, the (FGCB) having the geometrical characteristics shown in Figure 1 and 3 is examined. It is assumed that the beam is made of ceramic and metal, and that its material properties, i.e., Young's modulus *E* and mass density ρ, are functionally graded in the thickness direction (see equations 1 to 4).

Using the volume fractions V_c *and* V_m of the ceramic and metal constituents, a material property P can be expressed as:

$$P = P_m V_m + P_c V_c \tag{1}$$

where subscripts "*m*" and "*c*" refer to the metal and ceramic constituents, respectively. Various types of functions are used in the literature to describe the variation in the volume fraction of the constituents. Here, a power law is adopted as follows:

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^k \quad \text{with} \quad V_m + V_c = 1 \tag{2}$$

where *k* is a non-negative parameter (power-law exponent) which dictates the material variation profile through the thickness of the beam, shown in Figure 3. Effective material properties of the FG beam such as Young's modulus (*E*(z)), the mass density (ρ(z)) or Poisson's modulus (υ(z)) can be determined by substituting (2) into (1), which gives:

$$E(z) = E_m + (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{3}$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{4}$$

$$\nu(z) = \nu_m + (\nu_c - \nu_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{5}$$

where E_m and E_c are the Young's modulus of the lower surface (z = -*h*/ 2) and of the surface (z = *h*/2) of the FGM beam, the variation of Young's modulus in the thickness direction of the beam *k*-FGM beam is shown Figure 3 [5]. The volume fraction changes rapidly near the lower surface for *k* <1, and increases rapidly near the upper surface for *k* > 1.

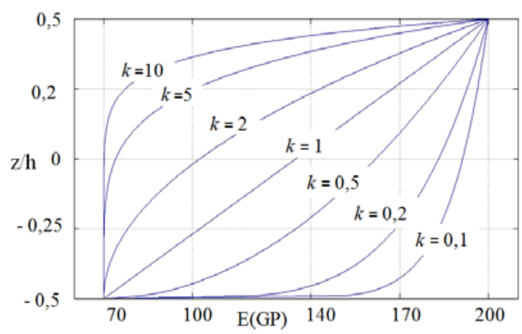

Fig. 3. Variation of the material properties of the FG beam through the thickness (Young Modulus)

3. THE HOMOGENIZATION PROCEDURE

The nonlinear strain–displacement relationships in a vibrating beam are given by:

$$\varepsilon_x^a = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad (6)$$

$$K_x = \frac{\partial^2 w}{\partial x^2} \quad (7)$$

where ε_x^a is the nonlinear axial strain and K_x is the beam curvature. The total elastic strain energy V_T of Euler-Bernoulli beams is:

$$V_T = \frac{1}{2}\int_0^L (N_x \varepsilon_x^a + M_y K_x)dx \quad (8)$$

In which N_x and M_y are respectively the axial internal force and the bending moment, related to strains as follows:

$$N_x = b\,A_{11}\varepsilon_x^a + b\,B_{11}K_x \quad (9)$$

$$M_y = b\,B_{11}\varepsilon_x^a + b\,D_{11}K_x \quad (10)$$

b: is the width of the beam

Where A_{11}, B_{11} and D_{11} are the extension-extension, the bending-extension and the bending-bending coupling coefficients respectively. These coefficients are evaluated using the classical functionally graded beam theory, which leads to the following expressions for the total strain energyV_T, in terms of the transverse displacement W [15] and [9]:

$$V_T = \frac{(ES)_{eff}}{8l}\left(\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right)^2 + \frac{(EI)_{eff}}{2}\int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \quad (11)$$

Where $(ES)_{eff} = bA_{11}$ and $(EI)_{eff} = b(D_{11} + (B_{11}^2/A_{11}))$ are the effective axial and bending stiffness respectively, A_{11}, B_{11} and D_{11} are given by:

$$(A_{11}, B_{11}, D_{11}) = \int_{-h/2}^{h/2} \frac{E(z)}{1-\upsilon(z)^2}(1, z, z^2)\,\mathrm{d}z \quad (12)$$

The expression for the total strain energy obtained is effective for replacing the (FGCB) problem with that of an equivalent isotropic beam problem [17].

4 DISCRETE MODEL, HAMILTON'S PRINCIPLE AND EXPLICIT METHOD

As stated above, the present approach consists on replacing the (FGCB) by an equivalent homogeneous beam. The characteristics of the equivalent beam have been calculated in section (3), allowing application of the semi analytical model developed previously to nonlinear structural vibration (see for example references [15] and [16]) involving three tensors, namely the mass tensor m_{ij}, the linear rigidity tensor k_{ij} and the nonlinearity tensor b_{ijkl}. By application of Hamilton's principle and spectral analysis, the nonlinear vibration problem, reduced to a nonlinear algebraic system, is solved for increasing numbers of dof. The present section is devoted to the application of a discretization procedure, similar to that used in [11] in order to express the general terms of the three tensors mentioned above (for more calculation details, please see equations (28) to (30) and (56) in reference [11]). In the following subsections, the general expressions for the general terms of the mass, linear and nonlinear rigidity tensors are given. Then, details are given concerning the application of Hamilton's principle and the method of solution of the nonlinear algebraic system is presented. The results are discussed in Section (5).

4.1 Expressions for the general terms of the mass, linear and nonlinear rigidity tensors

The coefficients of the mass matrix are written as:

$$m_{ij} = \delta_{ij} m \quad i, j = 1, ..., N \quad (13)$$

The longitudinal and torsional spring stiffness was calculated in [11]:

$$k = \frac{(ES)_{eff}}{l} \quad (14)$$

$$C = \frac{(EI)_{eff}}{l} = (N+1)\frac{(EI)_{eff}}{L} \quad (15)$$

The rigidity matrix [K_{ccN}] is symmetric and given by:

$$[K_{ccN}] = \frac{(EI)_{eff}}{l^3}\begin{pmatrix} 6 & -4 & 1 & 0 & 0 & . & . & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & . & . & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & . & . & . \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & . & . \\ 0 & 0 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & 1 & -4 & 6 & -4 & 1 \\ . & . & . & . & . & 0 & 1 & -4 & 6 & -4 \\ 0 & . & . & . & . & 0 & 0 & 1 & -4 & 6 \end{pmatrix} \quad (16)$$

The expressions for the components of the nonlinear rigidity tensor b_{ijkl} are:

$$b_{iiii} \cong \frac{(ES)_{eff}}{8l^3} \cong \frac{(ES)_{eff}}{8l^3} \quad i = 1, ..., N \quad (17)$$

$$b_{i(i-1)(i-1)(i-1)} = b_{(i-1)i(i-1)(i-1)} = b_{(i-1)(i-1)i(i-1)} = b_{(i-1)(i-1)(i-1)i} \cong -\frac{(ES)_{eff}}{8l^3}$$
$$i = 2, ..., N \quad (18)$$

$$b_{i(i-1)(i-1)i} = b_{(i-1)i(i-1)i} = b_{(i-1)(i-1)ii} = b_{i(i-1)i(i-1)} = b_{(i-1)ii(i-1)} = b_{ii(i-1)(i-1)} \cong \frac{(ES)_{eff}}{8l^3}$$
$$i = 2, ..., N \quad (19)$$

$$b_{iii(i-1)} = b_{ii(i-1)i} = b_{i(i-1)ii} = b_{(i-1)iii} \cong -\frac{(ES)_{eff}}{8l^3}$$
$$i = 2, ..., N \quad (20)$$

The other values are equal to zero. The solution process is identical to that used in [11].

4.2 Hamilton's principle

Hamilton's principle is a general variational principle allowing the equations of motion of any mechanical system to be obtained. For a conservative system, it is symbolically written as:

$$\delta \int_0^{2\pi/\omega} (V_T - T)dt = 0 \qquad (21)$$

where V_T and T are the system strain and kinetic energy discretized expressions obtained by writing the transverse displacement w_i of mass i as:

$$w_i = A_i \cos\left(\omega_{\text{disc}}^{\text{nl}} t\right) = a_j \Phi_{ij} \cos\left(\omega_{\text{disc}}^{\text{nl}} t\right) \qquad (22)$$

where A_i is the modulus of the displacement w_i expressed in the displacement basis DB, and a_i is the modulus of the displacement w_i expressed in the modal basis MB.

Replacing T and V_T (V_T=V_l+V_{nl}) in this equation by their expressions given in [11], integrating the time functions over a period of vibration, and calculating the derivatives of the function obtained with respect to the a_i's, leads to the following set of nonlinear algebraic equations in MB [15]:

$$3a_i a_j a_k \overline{b}_{ijkr} + 2a_i \overline{k}_{ir} - 2a_i (\omega_{disc}^{nl})^2 \overline{m}_{ir} = 0 \quad i, j, k, r = 1, \text{N} \qquad (23)$$

which can be written in a matrix form as:

$$3\left[B(a)\right]\{a\} + 2\left[\overline{K}\right]\{a\} - 2(\omega_{disc}^{nl})^2 \left[\overline{M}\right]\{a\} = \{0\} \qquad (24)$$

4.3 Explicit method

A general presentation of the method of solution of the algebraic system of equations (23) is presented in [11]. The formulation is based on an approximation which consists on assuming, when the first nonlinear mode shape is under examination, that the contribution vector $\{a\}^T = [a_1\, a_2 \ldots\ldots a_N]$ can be written as $\{a\}^T = [a_1\, \varepsilon_2 \ldots \varepsilon_i \ldots \varepsilon_N]$, with ε_i for $i=2,\ldots\ldots,N$ are small compared to a_1. Neglecting in the expression $a_i a_j a_k \overline{b}_{ijkr}$ of equation (23), which involves summation for the repeated indices *i, j, k* over the range $\{1,2,\ldots,N\}$, the first, the second and third order terms with respect to ε_s, i.e., terms of the type $a_1^2 \varepsilon_v \overline{b}_{11vr}$, of the type $a_1 \varepsilon_i \varepsilon_j \overline{b}_{1ijr}$ or of the type $\varepsilon_i \varepsilon_j \varepsilon_k \overline{b}_{ijkr}$, so that the only remaining term is $a_1^3 \overline{b}_{111r}$, leads to :

$$\left(\overline{k_{jr}} - (\omega_{\text{disc}}^{\text{nl}})^2 \overline{m_{jr}}\right)\varepsilon_j + \frac{3}{2} a_{1disc}^3 \overline{b}_{111r} = 0 \quad \text{for } r = 2,\ldots,N \qquad (25)$$

The contributions $\{a\}^T = \left[a_{1\text{disc}}\ \varepsilon_1\, \varepsilon_2 \ldots \varepsilon_N\right]$ in the modal basis are calculated explicitly in [16]

$$\varepsilon_r = \frac{\frac{3}{2} a_{1\text{disc}}^3 \overline{b_{111r}}}{((\omega_{disc}^{nl})^2 \overline{m_{rr}} - \overline{k_{rr}})} \quad \text{for } r = 2,\ldots,N \qquad (26)$$

Equations (26) are useful for analyzing the amplitude dependence of the nonlinear mode shape. When only the amplitude dependence of the nonlinear frequencies ω_{disc}^{nl} is of a main concern, it may be well estimated, as shown in [16] for an interesting range of vibration amplitudes, by the single mode approach, applied in the modal basis, which leads to:

$$\omega_{disc}^{nl} = \sqrt{\frac{\overline{k_{11}}}{\overline{m_{11}}}} \sqrt{1 + \frac{3}{2} \frac{\overline{b_{1111}}}{\overline{k_{11}}} a_{1disc}^2} \qquad (26)$$

where a_{1disc} is the contribution of the first linear mode to the fundamental nonlinear mode shape, written in the modal basis and used as a vibration parameter.

5. NUMERICAL RESULTS AND DISCUSSIONS

In the present work, the functionally graded beam is supposed to be made of the material considered in [17] (h = b = 0,1; L/h = 50). The top surface of the FG beam is ceramic rich Si_3N_4 (E_c=322.03 GPa, ρ_c=2370 kg/m^3, υ_c = 0,24), whereas the bottom surface of the FG beam is metal rich (E_m=207.08 GPa, ρ_m=8166 kg/m^3, υ_m = 0,3178). In Table 1, the intermediate parameters allowing the calculation of $\omega_{disc}^{nl\,*} / \omega_{disc}^{l\,*}$ are given ($R = \sqrt{(EI)_{eff} / (ES)_{eff}}$). In Tables 2 to 4, the fundamental nonlinear to linear frequency ratios $\omega_{disc}^{nl\,*} / \omega_{disc}^{l\,*}$ of the (FGCB) considered in the present numerical simulation, are given and compared with the published results in [9] and [17] for various vibration amplitudes.

Table 1: Intermediate parameters used in the calculation of $\omega_{nl}^* / \omega_l^*$

k	A_{11}	B_{11}	D_{11}	$(ES)_{eff}$	$(EI)_{eff}$	R	ρ
0,5	32,118	0,096	0,026	3,2118	0,0026	0,0284	4302
1	29,745	0,118	0,024	2,9745	0,0025	0,0289	5268
2	27,396	0,117	0,023	2,7396	0,0024	0,0295	6234

Table 2: Nonlinear to linear frequency ratios $\omega_{\text{nl}}^* / \omega_{\text{l}}^*$ of a FG clamped beam at various amplitudes $k = 0.5$

w^* (x* = 0.5)	Present	[9]	[17]
0	1.000	1.000	1.000
1	1.0175	1.023	1.056
2	1.0725	1.088	1.210
3	1.155	1.187	1.429
4	1.2675	1.313	1.689
5	1.395	1.457	1.974

Table 3: Nonlinear to linear frequency ratios $\omega_{\mathrm{nl}}^{*}/\omega_{\mathrm{l}}^{*}$ of a FG clamped beam at various amplitudes $k=1$

w^* (x* = 0.5)	Present	[9]	[17]
0	1.000	1.000	1.000
1	1.0225	1.023	1.056
2	1.0725	1.087	1.208
3	1.155	1.186	1.426
4	1.2675	1.311	1.685
5	1.395	1.454	1.968

Table 4: Nonlinear to linear frequency ratios $\omega_{\mathrm{nl}}^{*}/\omega_{\mathrm{l}}^{*}$ of a FG clamped beam at various amplitudes *k=2*

w^* (x* = 0.5)	Present	[9]	[17]
0	1.000	1.000	1.000
1	1.0197	1.022	1.055
2	1.0725	1.085	1.203
3	1.1575	1.181	1.417
4	1.2675	1.303	1.671
5	1.3925	1.443	1.949

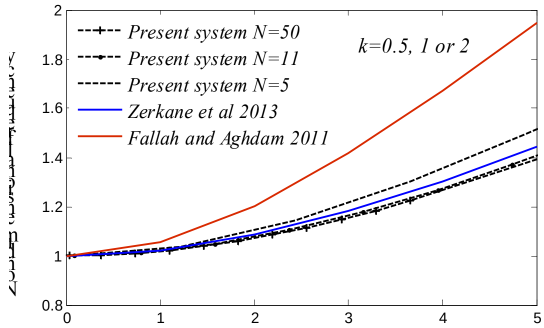

Fig. 4. Comparison between the backbone curves obtained via the discrete system and those based on the continuous theory for (FGCB) taken from references [9 and 17] in the case *k*=0.5, 1 or 2

It is noted in Figure 4 that the fundamental nonlinear frequency parameters found for the (FGCB) by the present discrete model are in a good agreement with previously published results in [9], based on the continuous theory for (FGCB), showing the validity of the equivalent discrete model for the FGCB. The solution given in [17] overestimates the frequencies of the (FGCB), especially for high values of the dimensionless amplitude, due to the assumptions made in the approximate solution procedure adopted. Examining Tables 2 to 4, it may be noted that the ratios of the (FGCB) nonlinear to linear frequency ratios do not change with *k*, while the nonlinear frequency itself depend on *k* as shown in Figure 5, which may be expected due to the effect of the constituent fractions on the beam global rigidity. Curves are given in Figure 4 for various values of the number *N* of dof used in the discrete model, i.e. *N*= 5, 11. It can be concluded that it is unnecessary to use a high number of dof since the model with 11 dof is good enough. The results obtained for the linear and nonlinear frequencies of the discrete system compare well with those based on the continuous beam theory.

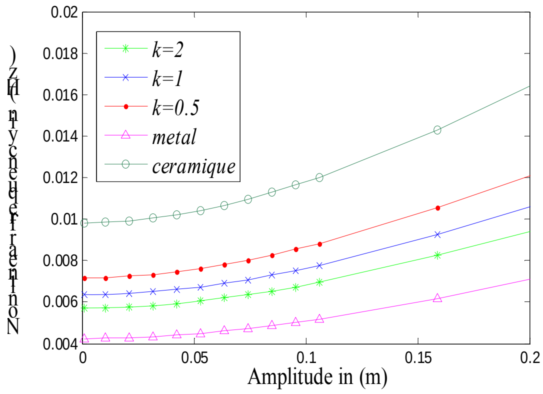

Figure 5: Comparison between the dimensional backbone curves obtained via the discrete system for (FGCB) for different values of the ceramic percentage.

6. CONCLUSION

The present study deals with the problem of geometrically nonlinear free vibrations of a (FGCB). Using Hamilton's principle and spectral analysis, and neglecting the axial inertia and damping, a homogenization procedure has been proposed which reduced the problem studied to that of an equivalent isotropic homogeneous beam with effective bending and axial stiffness parameters. Using a multimode approach, the nonlinear frequency ratios $\omega_{\mathrm{nl}}^{*}/\omega_{\mathrm{l}}^{*}$ calculated here have been compared to those given in [17] for isotropic and FG beams. An overestimate of the natural frequencies has been noted in the results given in [17], due to the approximations made in the solution adopted. On the other hand, the discrete model proposed in the present paper, shown in Figure 2, may be representative of the flexural vibration of a (FGCB) with various types of discontinuities. The values of the torsional stiffness C_2, C_3,…,C_{N+1} of springs 2, 3,…, N+1 and the masses m_1, m_2,…, m_N should be estimated in each case, depending on the type of singularities, such as point masses located at various positions of the beam [14] or a variation in the beam stiffness.

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