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LINEAR AND GEOMETRICALLY NON-LINEAR FREQUENCIES AND MODE SHAPES OF BEAMS CARRYING A POINT MASS AT VARIOUS LOCATIONS. AN ANALYTICAL APPROCH AND A PARAMETRIC STUDY

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Abstract

In the present paper, the frequencies and mode shapes of a clamped beam carrying a point mass, located at different positions, are investigated analytically and a parametric study is performed. The dynamic equation is written at two intervals of the beam span with the appropriate end and continuity conditions. After the necessary algebraic transformations, the generalised transcendental frequency equation is solved iteratively using the Newton Raphson method. Once the corresponding program is implemented, investigations are made of the changes in the beam frequencies and mode shapes for many values of the mass and mass location. Numerical results and plots are given for the clamped beam first and second frequencies and mode shapes corresponding to various added mass positions. The effect of the geometrical non-linearity is then examined using a single mode approach in order to obtain the corresponding backbone curves giving the amplitude dependent non-linear frequencies.

Keywords: Non-linear vibration, Hamilton's principle, Newton-Raphson, backbone curve, second formulation

1. INTRODUCTION

Studying the vibrations of beams supporting point masses is not a new topic. Indeed, many researchers have studied this problem in the linear regime and have presented results both analytically and experimentally [1 to 10]. In the present work, the first purpose was the development of a parametric model enabling engineers and designers, when dimensioning a machine or a bridge for example, to easily choose the value and the position of the mass which may added in order to control the frequency and avoid undesirable resonances. The second objective was the investigation of the effect of the geometrical nonlinearity induced by large vibration amplitudes on the amplitude dependent nonlinear frequencies, mode shapes and curvatures of "the clamped beam with an added point mass", denoted in what follows as CBAPM. The first part of the article deals with the linear case for a clamped beam supporting one point mass, for which the results obtained are summarized in tables or graphs, for various values of the mass and mass location. In the second part, the effect of the geometrical non-linearity is investigated, in the neighbourhood of the first and second modes, by combining in each case the CBAPM linear mode examined, calculated in the first part, with the non-linear single mode approach, based on the semi analytical method for non-linear structural vibrations developed previously [12]. This allowed various backbone curves to be drawn, corresponding to various values of the mass and mass location.

The multi-mode approach is then used, in two cases, to check the amplitude dependence of the non-linear modes and associated curvatures.

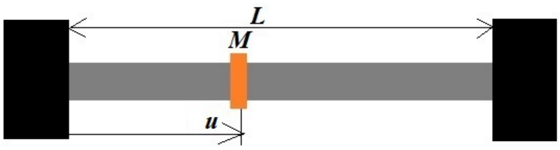

Fig. 1. The clamped beam with a point mass

2. FREE VIBRATIONS OF A CBAPM (LINEAR CASE)

2.1. Theoretical formulation

The uniform beam, clamped at both ends, with a point mass m, shown in fig.1, is made of a material of mass density ρ, Young's modulus E, length L, cross-sectional area S, radius of gyration r and second moment of area of cross section I. Let x be the coordinate along the beam neutral axis measured from the right end, u the coordinate of the added mass position, and $W(x,t)$ the beam transverse deflection, measured from its equilibrium position. Assuming harmonic motion, the beam transverse displacement may be expressed as:

$$W(x,t) = w(x)\sin(\omega t) \quad (1)$$

The free vibration of the CBAPM is governed by the following differential equation:

$$\frac{d^4w}{dx^4} - \beta^4 w = 0 \quad (2)$$

With:

$$\beta^4 = \frac{\rho S}{EI}\omega^2 \quad (3)$$

Introducing the non-dimensional coordinate $\eta = \frac{x}{L}$, and $\xi = \frac{u}{L}$, the beam transverse displacement function w may be defined in piecewise by:

$$w(\eta) = \begin{cases} w_1(\eta) \to \left]0, \xi\right[. \\ w_2(\eta) \to \left]\xi, 1\right[\end{cases} \quad (4)$$

The general solution for transverse vibrations in the first and second span, can be written as:

$$\begin{aligned} w_1(\eta) = a_1 \cosh(\beta_i L\eta) + b_1 \sinh(\beta_i L\eta) + \\ c_1 \cos(\beta_i L\eta) + d_1 \sin(\beta_i L\eta) \end{aligned} \quad (5)$$

$$\begin{aligned} w_2(\eta) = a_2 \cosh(\beta_i L(\eta - \xi)) + \\ b_2 \sinh(\beta_i L(\eta - \xi)) + \\ c_2 \cos(\beta_i L(\eta - \xi)) + d_2 \sin(\beta_i L(\eta - \xi)) \end{aligned} \quad (6)$$

in which

$$\beta_i = \sqrt[4]{\frac{\rho S \omega_i^2}{EI}} \quad (7)$$

for i= 1, 2, ... are the mode shape parameters of the CBAPM. The constants a_j, b_j, c_j, d_j are determined by the continuity and end conditions, as follows:

At the beam left clamped end:

$$w_{1i}(\eta)\big|_{\eta=0} = 0 \quad (8)$$

$$\left.\frac{dw_{1i}(\eta)}{d\eta}\right|_{\eta=0} = 0 \quad (9)$$

At the beam right clamped end:

$$w_{2i}(\eta)\big|_{\eta=1} = 0 \quad (10)$$

$$\left.\frac{dw_{2i}(\eta)}{d\eta}\right|_{\eta=1} = 0 \quad (11)$$

At the position ξ, the beam continuity equations:

$$w_{2i}(\eta)\big|_{\eta=\xi} = w_{1i}(\eta)\big|_{\eta=\xi} \quad (12)$$

$$\left.\frac{dw_{2i}}{d\eta}\right|_{\eta=\xi} = \left.\frac{dw_{1i}}{d\eta}\right|_{\eta=\xi} \quad (13)$$

$$\left.\frac{d^2w_{2i}}{d\eta^2}\right|_{\eta=\xi} = \left.\frac{d^2w_{1i}}{d\eta^2}\right|_{\eta=\xi} \quad (14)$$

$$\begin{aligned} \left.\frac{d^3w_{2i}}{d\eta^3}\right|_{\eta=\xi} = \left.\frac{d^3w_{1i}}{d\eta^3}\right|_{\eta=\xi} + \\ M(\beta_i L)^4 w_{1i}(\eta)\big|_{\eta=\xi} \end{aligned} \quad (15)$$

Where

$$M = \frac{m}{\rho S L} \quad (16)$$

Equations 8 to 11 constitutes a linear system with eight equations and eight unknowns. To avoid having only the trivial zero solution, the determinant of the system must vanish, which gives the frequency equation, leading to the linear frequencies and mode shapes of the CBAPM. The Newton–Raphson algorithm was used in the present paper to find the roots of the frequency equation, which are summarised in Tables1 and 2.

3. APPLICATIONS

3.1. A uniform beam with one point mass

A parametric study is performed in order to quantify the effect of both the added mass and its location on the CBAPM first and second frequencies and mode shapes, with their associated curvatures.

Table1: First eigenvalues of the CBAPM for different values of the mass and mass locations

	The added masses		
η	M=0.25	M=0.5	M=1
0	4.7300407	4.7300407	4.7300407
0.05	4.7292508	4.7284553	4.7268474
0.1	4.7192868	4.7081653	4.6848083
0.15	4.6845584	4.6368121	4.5364706
0.2	4.6148728	4.4994416	4.2820554
0.25	4.5167386	4.3252078	4.0152076
0.3	4.4100531	4.1576632	3.7951258
0.35	4.314193	4.0213819	3.6322703
0.4	4.2409346	3.9239959	3.5220348
0.45	4.1956826	3.8661884	3.4585011
0.5	4.1804404	3.8470713	3.4377627

Table 2: second eigenvalues of the CBAPM for different values of the mass and mass locations

	The added masses		
η	M=0.25	M=0.5	M=1
0	7.8532046	7.8532046	7.8532046
0.05	7.844187	7.8347451	7.8145012
0.1	7.7426364	7.6164936	7.3377608
0.15	7.4876528	7.1496358	6.6519083
0.2	7.2372484	6.8350764	6.3987003
0.25	7.1313711	6.7751212	6.4487924
0.3	7.1747075	6.8938119	6.6557936
0.35	7.3280757	7.1251253	6.9564789
0.4	7.5448333	7.4213136	7.315858
0.45	7.757099	7.7137368	7.6737227
0.5	7.8532046	7.8532046	7.8532046

Figure 2 shows the variation in the first and second frequencies due to the presence of an added mass located at a position u, when u varies from 0 to 1. It can be seen that the fundamental frequency is affected, as may be expected, by the presence of the added mass in the vicinity of the beam centre, while the second frequency is affected by a mass present around u=L/4. These results are summarised in Tables 1 and 2 for three values of the added mass m. Figures 3 to 6 show the effect of the added mass on the clamped beam normalized first and second mode shapes and the associated curvatures, which are related to the beam bending moments and stresses, for various mass locations and for various values of the mass located at u=L/2 and u=L/4 respectively.

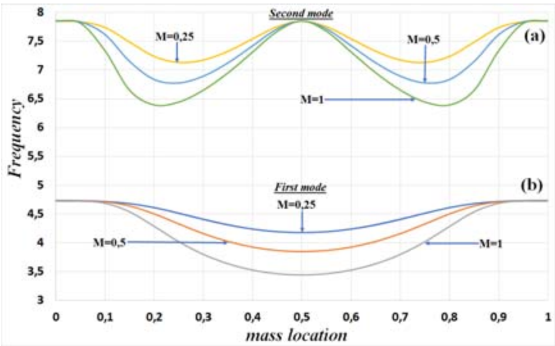

Fig. 2 Effect of the added mass location on the linear frequencies associated to (a) the first mode (b) the second mode, for M = 0.25, 0.5 and 1

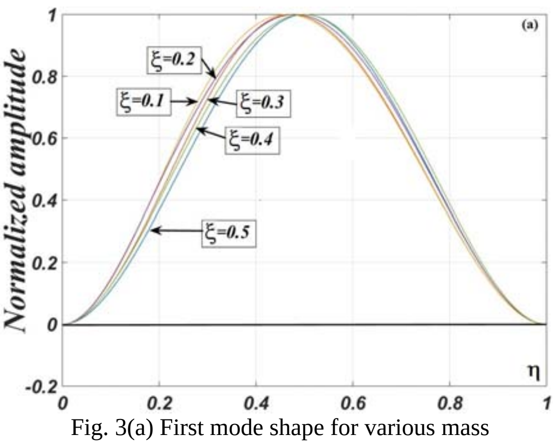

Fig. 3(a) First mode shape for various mass locations for M = 0.5.

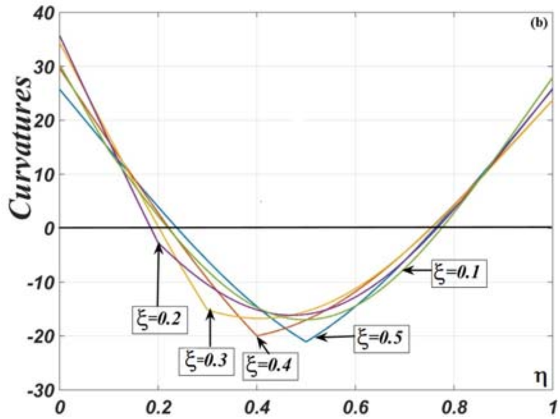

Fig. 3(b) Curvatures associated to the first mode shape for M = 0.5 at various locations.

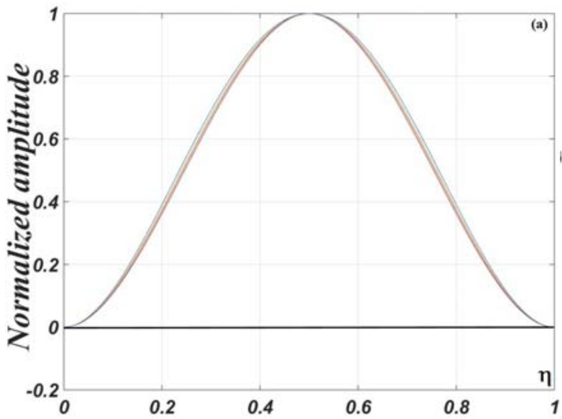

Fig. 4(a) First normalized mode shape for various values of the mass located at the middle of the beam (u=L/2).

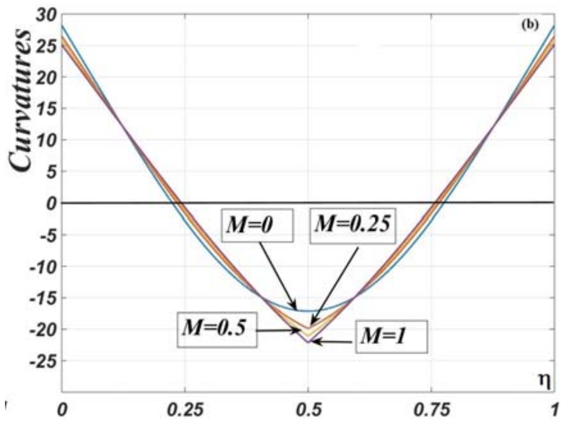

Fig. 4(b) Curvatures associated to the first mode for various values of the mass located at the middle of the beam (u=L/2).

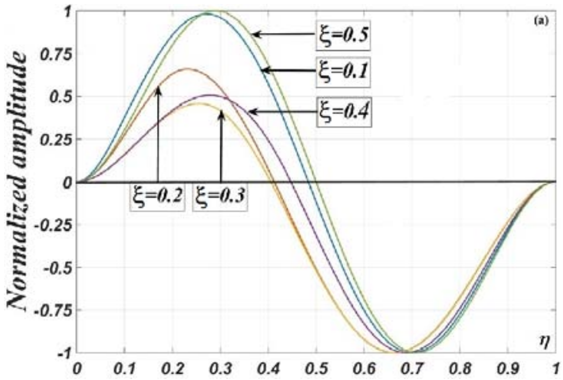

Fig. 5(a) Second mode shape for various mass locations and M=0.5

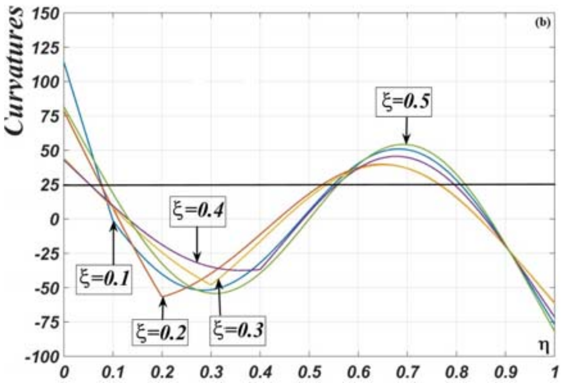

Fig. 5(b) Curvatures associated to the second mode for various mass locations and M=0.5

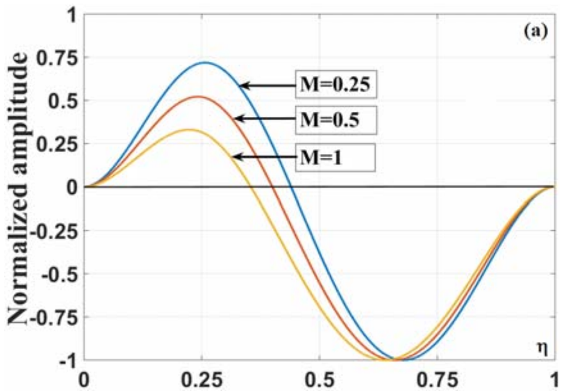

Fig. 6(a) Second normalized linear mode shape for various values of the mass located at the quarter of the beam (u=L/4)

4. GEOMETRICALLY NON-LINEAR VIBRATION OF A CBAPM

The total kinetic energy of the CBAPM can be expressed as:

$$T = \frac{1}{2}\rho S\int_0^L \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}m\left(\frac{\partial w(x_j,t)}{\partial t}\right)^2 \quad (17)$$

The beam total strain energy can be written as the sum of the strain energy due to the bending denoted as V_{lin}, plus the axial strain energy due to the axial load induced by the large deflections de In which ∂ indicates the variation of the integral. Introducing the assumed series (20) into the energy condition (27) via equations (21) to (23) reduces the problem to that of finding the minimum of the function Φ given by:noted as V_{Nlin} [12-13].

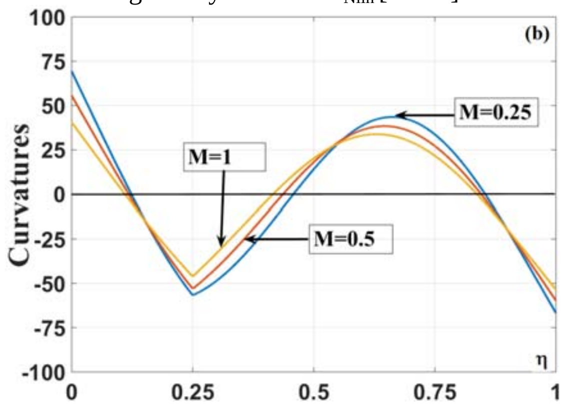

Fig. 6(b) Curvatures associated to the second mode for various values of the mass located at the quarter of the beam (u=L/4)

$$V_{Lin} = \frac{1}{2}EI\int_0^L \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx \quad (18)$$

$$V_{Nlin} = \frac{1}{8}\frac{ES}{L}\left[\int_0^L \left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx\right]^2 \quad (19)$$

To develop the non-linear theory, the transverse displacement function is expanded as a series of N basic spatial functions (the CBAPM linear modes calculated above):

$$w(x,t) = q_i(t)w_i(x) = a_i w_i \sin(\omega t) \quad (20)$$

Where the usual summation convention for repeated indices is used. One obtains after discretization of the expressions (17-19):

$$T = \frac{1}{2}\omega^2 a_i a_j \cos^2(\omega t) m_{ij} \quad (21)$$

$$V_{Lin} = \frac{1}{2}a_i a_j k_{ij} \sin^2(\omega t) \quad (22)$$

$$V_{NLin} = \frac{1}{2}a_i a_j a_k a_l b_{ijkl} \sin^4(\omega t) \quad (23)$$

With

$$m_{ij} = \rho S\int_0^L w_i w_j dx + m w_i(u)\times w_j(u) \quad (24)$$

$$k_{ij} = EI\int_0^L \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_j}{\partial x^2}dx \quad (25)$$

$$b_{ijkl} = \frac{1}{4}\frac{ES}{L}\int_0^L \frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}dx\int_0^L \frac{\partial w_k}{\partial x}\frac{\partial w_l}{\partial x}dx \quad (26)$$

The dynamic behavior of the CBAPM is governed by Hamilton's principle, which is symbolically written as:

$$\partial\int_0^{\frac{2\pi}{\omega}} (V - T) = 0 \quad (27)$$

In which ∂ indicates the variation of the integral. Introducing the assumed series (20) into the energy condition (27) via equations (21) to (23) reduces the problem to that of finding the minimum of the function Φ given by:

$$\Phi = \int_0^{\frac{2\pi}{\omega}} \begin{bmatrix} \frac{1}{2} a_i a_j k_{ij} \sin^2(\omega t) + \\ \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4(\omega t) - \\ \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2(\omega t) \end{bmatrix} dt \qquad (28)$$

with respect to the undetermined constant a_i. Integrating the trigonometric functions $\sin^2(\omega t)$, $\sin^4(\omega t)$ and $\cos^2(\omega t)$ over a period of vibration leads to the following expression:

$$\Phi = \frac{\pi}{2\omega} \begin{pmatrix} a_i a_j k_{ij} + \frac{3}{4} a_i a_j a_k a_l b_{ijkl} - \\ \omega^2 a_i a_j m_{ij} \end{pmatrix} \qquad (29)$$

Φ appears in Equation (29) as a function of the undetermined constant, a_i, i=1, …, N. Equation (27) reduces to:

$$\frac{\partial \Phi}{\partial a_r} = 0 \qquad r = 1, 2, \dots, N \qquad (30)$$

Generally, and this is the case for all of the applications previously made of the present theory, the tensors k_{ij} and m_{ij} are symmetric, and the tensor b_{ijkl} is such that: $b_{ijkl} = b_{klij}$, $b_{ijkl} = b_{jikl}$. Taking into account these properties of symmetry, it appears that equations (30) are equivalent to the following nonlinear algebraic system:

$$2a_i k_{ir} + 3a_i a_j a_k b_{ijkr} - 2\omega^2 a_i m_{ir} = 0 \qquad (31)$$

Putting $b_{ij}(\{A\}) = a_k a_l b_{ijkl}$, the nonlinear geometrical rigidity matrix [B] is defined. Each term of matrix [B] is a quadratic function of the column matrix of coefficients $\{A\} = [a_1 a_2 \dots a_n]^T$. Introducing matrix [B] in equations (31) leads to the following matrix equation:

$$2[K]\{A\} + 3[B(\{A\})]\{A\} = 2\omega^2 [M]\{A\} \qquad (32)$$

Where [K] and [M] are the classical rigidity and mass matrices respectively. It should be noticed that, by neglecting the nonlinear term, $[B(\{A\})]\{A\}$, equation (32) reduces to the classical eigenvalue problem:

$$[K]\{A\} = \omega^2 [M]\{A\} \qquad (33)$$

Equation (32) is an extension of the Rayleigh-Ritz formulation to the nonlinear case. It has to be solved numerically, or explicitly [12]. Finally, one may obtain, via simple transformations, the following nonlinear algebraic system [11-12]:

$$2a_i k_{ir} + 3a_i a_j a_k b_{ijkr} - 2\frac{a_i a_j k_{ij} + a_i a_j a_k a_l b_{ijkl}}{a_i a_j m_{ij}} a_i m_{ir} = 0 \qquad (34)$$

To obtain non-dimensional parameters, we put:

$$w(x) = r w_i^* \left(\frac{x}{L}\right) = r w_i^*(\eta) \qquad (35)$$

$$\frac{\omega^2}{\omega^{*2}} = \frac{EI}{\rho S L^4} \qquad (36)$$

$$\frac{m_{ij}}{m_{ij}^*} = \rho S L r^2 \qquad (37)$$

$$\frac{K_{ij}}{K_{ij}^*} = \frac{EIr^2}{L^3} \qquad (38)$$

$$\frac{b_{ijkl}}{b_{ijkl}^*} = \frac{EIr^2}{L^3} \qquad (39)$$

In which $k_{ij}^*, m_{ij}^*, b_{ijkl}^*$ are non-dimensional tensors given by:

$$m_{ij}^* = \int_0^1 w_i^* w_j^* d\eta + M w_i^*(\xi) w_j^*(\xi) \qquad (40)$$

$$k_{ij}^* = \int_0^1 \frac{\partial^2 w_i^*}{\partial \eta^2} \frac{\partial^2 w_j^*}{\partial \eta^2} d\eta \qquad (41)$$

$$b_{ijkl}^* = \alpha \int_0^1 \frac{\partial w_i^*}{\partial \eta} \frac{\partial w_j^*}{\partial \eta} d\eta \times \int_0^1 \frac{\partial w_k^*}{\partial \eta} \frac{\partial w_l^*}{\partial \eta} d\eta \qquad (42)$$

and α is the non-dimensional parameter:

$$\alpha = \frac{Sr^2}{4I} \qquad (44)$$

Substituting these equations into equation (34) leads to:

$$2a_i k_{ir}^* + 3a_i a_j a_k b_{ijkr}^* - 2\frac{a_i a_j k_{ij}^* + a_i a_j a_k a_l b_{ijkl}^*}{a_i a_j m_{ij}^*} a_i m_{ir}^* = 0 \qquad (45)$$

The non-linear algebraic system (45) can be solved using an iterative method as in [13] or explicitly with the so-called in [11] first and second formulations.

4.1. Single mode approach (SMA)

From equation (32), one can calculate the frequency ω by multiplying the two hand sides of the equation from the left by $\{A\}^T$, which gives:

$$\omega^2 = \frac{\{A\}^T [K]\{A\} + \frac{3}{2}\{A\}^T [B(\{A\})]\{A\}}{\{A\}^T [M]\{A\}} \qquad (46)$$

The (SMA) consists of neglecting all the basic function contributions except a single ''resonant'' mode in order to reduce the multi-degree-of-freedom system to a single one. The (SMA) is often used in the literature [12], due to the great simplification it introduces in the theory and because the error it introduces in the estimation of the amplitude dependent nonlinear frequencies remains small. Applying the (SMA) to equation (46) leads to:

$$\omega^2 = \frac{k}{m} + \frac{3}{2} a^2 \frac{b}{m} \qquad (47)$$

In which $k=k_{rr}$, $m=m_{rr}$ and $b=b_{rrrr}$ correspond to the single mode in the neighbourhood of which the non-linear effect is examined, i.e. the r^{th} linear mode calculated in section 3 of the CBAPM. Figures 7 and 8 give the various backbone curves corresponding to various values of the mass and mass location for the CBAPM first and second non-linear modes, corresponding r=1 and r=2 respectively. It can be seen that the added mass accentuates the non-linear hardening effect when its location is closer to u =L/2 for the first nonlinear mode and u=L/4 for the second nonlinear mode. On the other hand, for a given location of the added mass, increasing the mass leads to a reduction in the hardening effect, which can be understood, by considering equation (47).

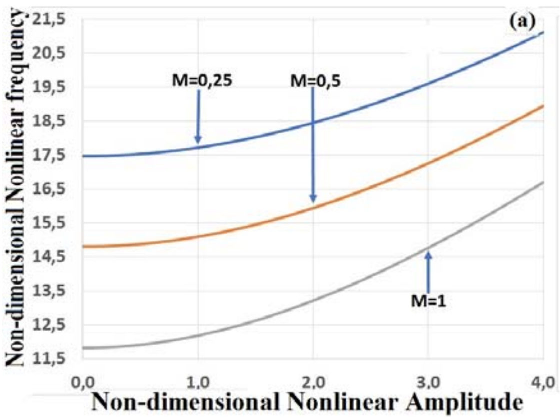

Fig. 7(a) First nonlinear mode Backbone curves for various values of the mass located at the middle

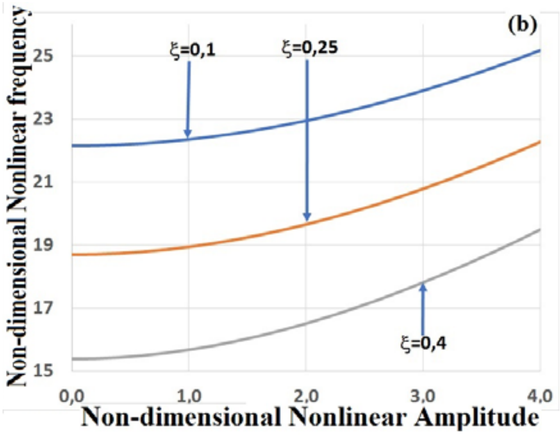

Fig. 7(b) First nonlinear mode Backbone curves for various locations of the mass M=0.5

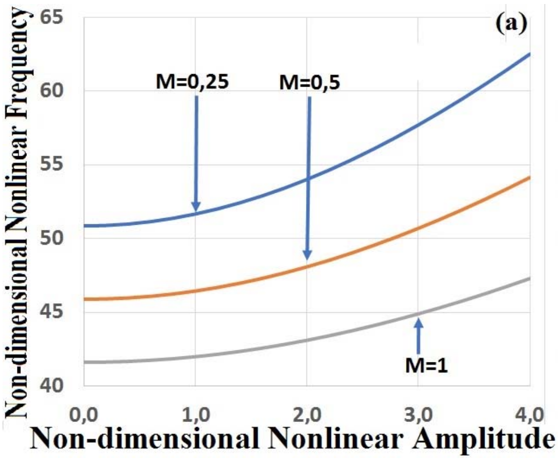

Fig. 8(a) Second nonlinear mode Backbone curves for various values of the mass located at the middle

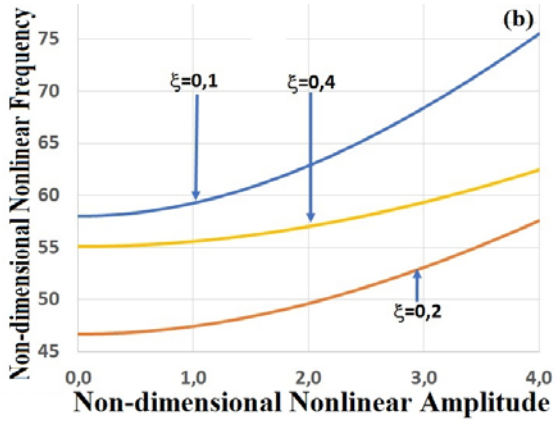

Fig. 7(b) second nonlinear mode Backbone curves for various locations of the mass M=0.5

4.2. Multi-mode approach (MMA)

A multimode analysis is made in the present section to non-linear vibrations of two clamped beams with an added mass M=0.5 located at ξ=0.5 and ξ=0.25 denoted as CBA0.5PM and CBA0.25PM respectively. In each case, 10 linear modes are determined using the parameter β_i, computed numerically using the standard Newton-Raphson iterations and summarised in Tables 3 and 5. The modes obtained have been normalized in such a manner that the corresponding mass matrix equals the identity matrix. Figures 8 and 9 shows the symmetrical and anti-symmetrical linear mode shapes used in the non-linear analysis as basic functions for i=1 to 10, M=0.5 and ξ=0.5. Figures 12 and 13 shows in the same way the symmetrical and anti-symmetrical linear mode shapes used in the non-linear analysis as basic functions for i=1 to 10, M=0.5 and ξ=0.25.

The parameters m^*_{ij}, k^*_{ij} and b^*_{ijkl} of equations (40 to 42) were computed numerically by using Simpson's rule in the range [0,1]. The non-linear algebraic system (45) has been solved using the second formulation [11], which can be explained as follows:

The basic approximation behind this formulation consists on writing the contribution vector to non-linear mode as: $\{A\}=[a_1,\varepsilon_3,...,\varepsilon_{10}]$ and neglecting in the expression $a_i a_j a_k b_{ijkr}$ of equations (45) second order and third order terms with respect to e_i, i.e., terms of the type $a_1\varepsilon_j\varepsilon_k b_{1jkr}$ and $\varepsilon_i\varepsilon_j\varepsilon_k b_{ijkr}$.

Table 3: Symmetric and anti-symmetric eigenvalues parameters (M=0,5 and ξ=0,5)

$\beta_i L$				
i	Symmetric		i	Anti-symmetric
1	3.847071303		2	7.853204624
3	9.999905785		4	14.13716549
5	16.09984075		6	20.42035217
7	22.29314344		8	26.70365446
9	28.51961166		10	32.96211166

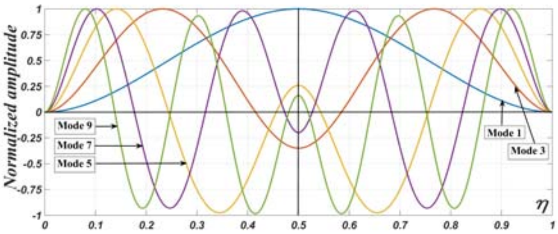

Fig. 8. Symmetrical CBAPM functions for i=1,3,5,7,9 (M=0.5 and ξ=0.5)

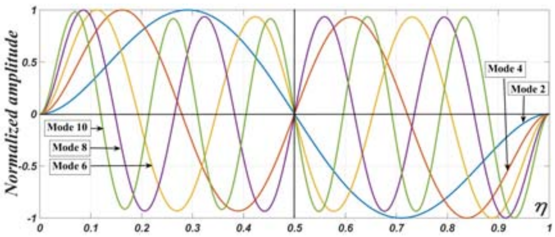

Fig. 9. Anti-symmetrical CBA05PM linear mode shapes for i=2,4,6,8,10 (M=0,5 and ξ=0,5)

This permits one to approximate the non-linear algebraic system (45) by an equivalent linear system, as detailed in [11]. The CBA0.5PM basic functions contributions obtained by this method for a_1=1, 4, 6, and 10 in the case of M=0.5 and ξ=0.5 are summarised in Table 4. The corresponding normalised non-linear modes and curvatures are shown in Figures 10 and 11 respectively. It can be noticed in table 4 that the contributions of antisymmetric functions are, as may be expected, very small compared to those of the symmetric functions. Also, Figures 10 and 11 exhibit the amplitude dependence of the CBA0.5PM first non-linear mode shape and associated curvatures, which show that the (SMA), although it leads to good estimates of non-linear frequencies, may underestimate the non-linear stresses in the beam examined and justifies, when the stresses are of the main interest, use of the (MMA). The CBA0.25PM basic functions contributions obtained by this method for a_1=1, 4, 6, and 10 in the case of M=0.5 and ξ=0.25 are summarised in Table 5. The corresponding normalised non-linear modes and curvatures are shown in Figures 14 and 15 respectively. The results obtained confirm the conclusions mentioned above, related to the use of (SMA) and (MMA).

	Table 4. Basic functions coefficient contribution			
a_1	1.00E+00	4.00E+00	6.00E+00	1.00E+01
a_2	-4.08E-05	-1.52E-04	-2.07E-04	-2.54E-04
a_3	-2.64E-04	-1.60E-02	-5.02E-02	-1.91E-01
a_4	3.18E-07	1.96E-05	6.30E-05	2.54E-04
a_5	-1.43E-04	-8.86E-03	-2.87E-02	-1.18E-01
a_6	-5.37E-09	-2.15E-07	-2.30E-07	4.12E-06
a_7	-4.86E-06	-3.35E-04	-1.22E-03	-6.45E-03
a_8	3.41E-08	2.20E-06	7.50E-06	3.53E-05
a_9	9.72E-06	6.26E-04	2.12E-03	9.90E-03
a_{10}	4.42E-08	2.82E-06	9.49E-06	4.35E-05

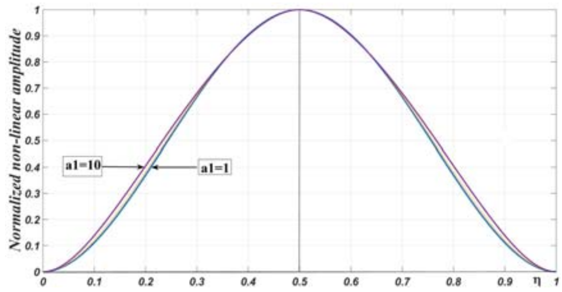

Fig. 10. The normalized first non-linear mode shape of a CBA05PM (M=0.5 and ξ=0.5)

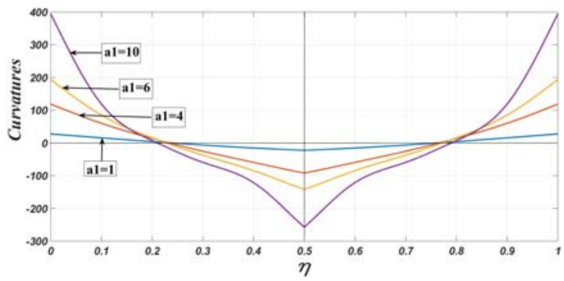

Fig. 11. The first non-linear curvatures of the CBAPM (M=0.5 and ξ=0.5)

Table. 5 summarise the symmetric and anti-symmetric eigenvalues in the case M=0.5 and ξ=0.25.

Table 5: Symmetric and anti-symmetric eigenvalues parameters (M=0.5 and ξ=0.25)

$\beta_i L$				
i	Symmetric		i	Anti-symmetric
1	4.32520777		2	6.77512116
3	10.2269194		4	13.97746152
5	17.0516317		6	19.21667971
7	22.7158396		8	26.53359815
9	29.5559992		10	31.63599915

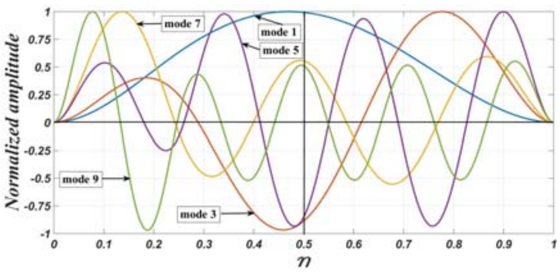

Fig. 12. Symmetrical CBA0.25PM functions for i=1,3,5,7,9 (M=0.5 and ξ=0.25)

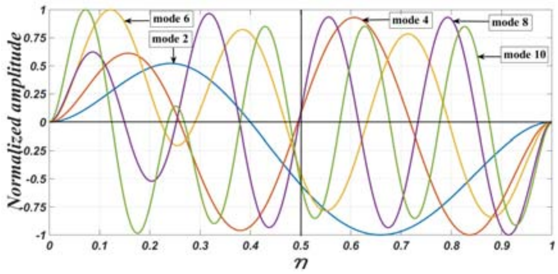

Fig. 13. Anti-symmetrical CBA0.25PM functions for i=2,4,6,8,10 (M=0.5 and ξ=0.25)

	Table 6. Coefficient contribution			
a_1	3.81E-03	1.80E-01	4.53E-01	1.16E+00
a_2	1.00E+00	4.00E+00	6.00E+00	1.00E+01
a_3	4.76E-04	2.72E-02	7.90E-02	2.44E-01
a_4	7.40E-04	4.51E-02	1.41E-01	5.13E-01
a_5	-5.99E-05	-3.88E-03	-1.30E-02	-5.68E-02
a_6	2.24E-05	1.47E-03	4.99E-03	2.24E-02
a_7	-6.46E-05	-4.22E-03	-1.43E-02	-6.27E-02
a_8	1.99E-05	1.33E-03	4.57E-03	2.12E-02
a_9	-1.13E-05	-7.61E-04	-2.66E-03	-1.26E-02
a_{10}	-2.91E-05	-1.95E-03	-6.78E-03	-3.18E-02

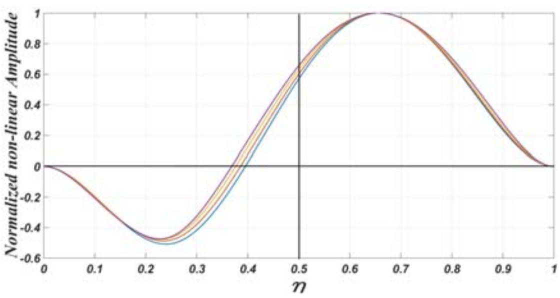

Fig. 14. The normalized non-linear second mode shapes (M=0.5 and ξ=0.25)

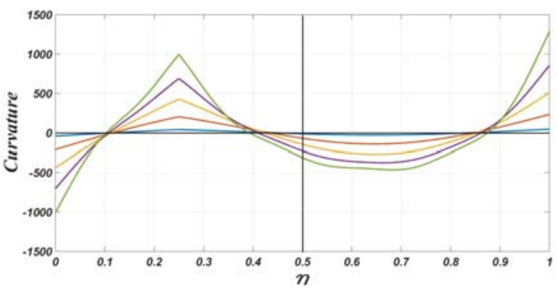

Fig. 15. The second non-linear curvatures of the CBAPM (M=0.5 and ξ=0.25)

5. ONCLUSION

A parametric study has been performed and results have been quantified for the effects of an added point mass, at various beam locations, on the frequencies and the first two linear modes of the new system. On the other hand, investigations were carried out in order to determine quantitatively how far do "beam with an added point mass" frequencies deviate from the linear ones, when the geometrical nonlinearity is taken into account. Using the linear modes calculated in the first part of the paper and a single mode approach (SMA), various backbone curves have been obtained, corresponding to different values of the added mass, and added mass location.

The theoretical formulation of the multimode approach (MMA) has then been presented and numerical solutions have been determined using the approximate procedure, defined in [11] and labelled second formulation. The results show the amplitude dependence of the non-linear modes and associated curvatures in all of the cases examined. This confirms that the linear theories may underestimate the induces stresses when large vibration amplitudes are involved.

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