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LAPLACE TRANSFORM FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN VISCOELASTIC ELECTRICAL SYSTEMS

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Abstract

This paper studies the use of the Laplace transform as a key tool for solving fractional differential equations which involve non-integer derivatives and are used to model various physical phenomena such as viscoelastic materials and control systems Fractional differential equations pose significant challenges due to the complexity of fractional derivatives and integral terms making classical solution methods inefficient The methodology in the paper relies on the Laplace transform to convert fractional equations shifting to a frequency domain to the temporal domain simplifying the handling of these complex equations This approach enables precise and efficient solutions and transforms complex equations into more manageable forms The study also explores practical applications such as solving equations related to viscoelastic materials which exhibit dynamic behavior governed by fractional equations This contributes to a deeper understanding of these materials and their mathematical modeling The paper concludes that the Laplace transform offers a robust framework for solving a wide range of fractional differential equations more efficiently with significant benefits in mathematical modeling and analysis Additionally the study highlights the importance of integrating digital methods with the Laplace transform for solving complex boundary problems thereby enhancing practical applications in fields like applied mathematics and engineering

Keywords: Laplace, fractional, derivatives, transform, equations, modeling

1. INTRODUCTION

The Laplace transform, introduced by Pierre-Simon Laplace in 1782, is a key tool for solving linear differential equations [1], [2]. It converts a function shifting to a frequency domain to the temporal domain and is widely used in fields like control engineering, electrical circuits, and signal processing. Unlike the Fourier transform, which represents functions as vibrational modes, the Laplace transform resolves functions into their moments [3], [4], [5].

The application of Laplace transforms in solving fractional differential equations marks a significant advancement in mathematical analysis, offering new methods to tackle complex problems across various fields. These equations, involving both fractional derivatives and integral terms, appear in real-world phenomena such as viscoelastic materials and control systems. Laplace transforms simplify their solution by converting them Converting from the frequency spectrum to the time frame enables better analysis of temporal signals.

Key methodologies include the decomposition method, which Yang et al in [6] demonstrated as an effective approach for obtaining approximate solutions with high precision. By breaking down complex problems into simpler components, this method improves computational efficiency. Another approach is compliant fractional transformations, explored by Ozkan et al in [7], which integrate fractional derivatives within the Laplace transform framework to address partial equations with irregular orders.

The Yang-Laplace transform, used by Jassim in [8], provides exact solutions for Volterra integro-differential equations by combining local fractional operators with Laplace transforms. This method enhances understanding of physical phenomena modeled by such equations. Additionally, Zada, A et al in [9] showcased the robustness of the ρ-Laplace transform in solving Liouville-Caputo fractional equations, highlighting its practical applications.

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Lastly, Shah et al in [10] introduced digital techniques using Laplace transforms for solving complex boundary problems, emphasizing the growing role of digital methods in conjunction with Laplace transformations in applied mathematics.

In this context, the study presented in this paper explores the mathematical framework of Laplace transforms, highlighting they're their practical applications in solving differential equations, convolution operations, and their impact in fields such as control engineering and signal processing.

2. PROBLEM AND METHODOLOGY

2.1 Problem

The paper addresses the challenge of solving fractional differential equations, which involve non-integer derivatives and are essential for modeling various real-world phenomena such as viscoelastic materials and control systems. Solving these equations is challenging. using classical methods, and their complexity lies in the presence of fractional derivatives and integral terms that require advanced mathematical techniques for simplification.

a. Methodology

The Laplace transform is employed as the primary method to simplify and solve fractional differential equations. By converting functions Converting from the frequency spectrum to the time frame enables better analysis of temporal signals., the Laplace transform makes it easier to handle complex differential equations. This transformation allows for the solution of fractional equations by transforming them into a more manageable form in the frequency domain. The methodology not only simplifies the equations but also provides a generalized framework for applying the Laplace transform to a broad range of problems across various fields such as control engineering and signal processing.

3. LT FOR FRACTIONAL DIFFERENTIAL EQUATIONS

In this part of the paper, we will cover some essential preliminaries, which are summarized below

a. The causal function for a fractional derivative h(t) is defined as:

$$\frac{d^\alpha}{dt^\alpha}h(t) = \begin{cases} h^{(m)}(t) & If\ \alpha = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\alpha)}\int_0^\beta \frac{h^{(m)}(t)}{(t-x)^{\alpha-m+1}}dt & If\ m-1<\alpha<m \end{cases} \quad (1)$$

The Euler gamma's function, Γ (.) Can be obtained as follows:

$$\Gamma(u) = \int_0^\infty t^{(u-1)}.e^{-t}\,dt \quad where\ \mathbb{R} > 0 \quad (2)$$

b. The LT of h(t), $t \in [0, \infty]$ It's determined by:

$$\mathcal{L}[h(t)](w) = \int_0^\infty e^{-wt}.h(t)dt\,, (w \in \mathbb{C}) \quad (3)$$

c. The Mittag-Leffler formula is expressed as follows: [11, 12]:

$$E_{\alpha,\sigma(u)} = \sum_{z=0}^{\infty}\frac{u^z}{\Gamma(\alpha z+\sigma)}, (u,\alpha,\sigma \in \mathbb{C}\,, \mathbb{R}(\sigma) > 0) \quad (4)$$

d. The Wright function that is the simplest [12], [13] is defined as:

$$\xi(\alpha,\sigma,u) = \sum_{z=0}^{\infty}\frac{1}{\Gamma(\alpha z+\sigma)}\frac{u^z}{z!}\,, (u,\alpha,\sigma \in \mathbb{C}\,) \quad (5)$$

e. The general Wright function ${}_n\Psi_m(u)$ is defined for u∈ ℂ,(ai,bj)∈ ℂ [30],[32],and real αi ,σj ∈ ℝ (i=1,….,n; j=1,….m) by the series :

$${}_n\Psi_m(u) = {}_n\Psi_m\left\{\begin{matrix}(a_i,\alpha_i)_{1,n} \\ (b_j,\sigma_j)_{1,m}\end{matrix}\middle| u\right\} = \sum_{z=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma(a_i+\alpha_i z)}{\prod_{j=1}^{m}\Gamma(b_j+\sigma_j z)}.\frac{u^z}{z!} \quad (6)$$

f. The fractional derivatives of Riemann-Liouville [33],[34] $D^\alpha_{a+}y$ and $D^\sigma_{b-}y$ of order $\alpha \in \mathbb{C}\,(\mathbb{R}(\alpha) \geq 0)$ are determined by:

$$D^\alpha_{a+}y(x) = \frac{1}{\Gamma(n-\alpha)}.\left[\frac{d.}{dx}\right]^n\int_0^x\frac{y(t)dt}{(x.-t)^{\alpha-n+1}}$$
$$(n = [\mathbb{R}(\alpha)] + 1; x > a) \quad (7)$$

$$D^\alpha_{a+}y(x.) = \frac{1}{\Gamma(n-\alpha)}\left[-\frac{d}{dx}\right]^n.\int_x^b\frac{y(t)dt}{(x.-t)^{\alpha-n+1}},$$
$$(n = ([\mathbb{R}(\alpha)] + 1, x < b) \quad (8)$$

Respectively, where $[\mathbb{R}(\alpha)]$ represents the integral part of $\mathbb{R}(\alpha)$

g. The shifted factorial [17], [18], with (1) *n= n!* for $n \in \mathbb{N}_0$ = (0,1, 2, . . .) is defined as:

$$(\boldsymbol{\delta})_{\boldsymbol{n}} = \begin{cases} 1 & (n=0), \\ \delta(\delta+1)\ldots(\delta+n-1) & (n \in \mathbb{N}_0/\{0\}\end{cases} \quad (9)$$

h. The definition of the binomial coefficients is:

$$\binom{\delta}{n} = \frac{\delta\,!}{\delta!\,(\delta-n)!} = \frac{\delta(\delta-1)(\delta-n+1)}{(n)!}, \quad (10)$$

Were.0! =1, and (δ, n) are integers then:

$$\binom{\delta}{0} = 1, \binom{\delta}{\delta} = 1, and\ (1-u)^{-\delta} = \sum_{q=0}^{+\infty}\frac{(\delta\,)_q}{q.!}.u^n = \sum_{q=0}^{\infty}\begin{bmatrix}\delta+(q-1) \\ q\end{bmatrix}.u^q \quad (11)$$

i. $\mathcal{L}[\xi(\alpha,\sigma;t)](.S) = (\frac{1}{S}).E_{\alpha,\sigma}(1/S)\ (\alpha > -1,\ \sigma \in \mathbb{C}\,; R(S) > 0)$ (12)

j. The LT of the generalized Wright function is:

$$\mathcal{L}\left\{{}_n\Psi_m\left[\begin{matrix}(a_i,\alpha_i)_{1,n} \\ (b_j,\sigma_j)_{1,m}\end{matrix}\middle| -t\right]\right\}(S) = \frac{1}{S}\,{}_{n+1}\Psi_m\left(\begin{matrix}(1,1);(a_i,\alpha_i)_{1,n} \\ (b_j,\sigma_j)_{1,m}\end{matrix}\middle| -\frac{1}{S}\right) \quad (13)$$

$(R(S) > 0)$, i = 1, … …. n and j = 1, … … m)

k. $\mathcal{L}[D^\beta h(t)](S) = S^\beta[\mathcal{L}h(t)](s) - \sum_{q=1}^{n}S^{\beta-q}h^{(q-1)}(0)$ (14)

Where β>0, n-1<β≤n, h (t) $\in C^n(0,\infty)$, $h^n(t) \in L_1(0,b))$ for any b>0) [19]

l. The inverse Laplace transform is:

$$\mathcal{L}^{-1}\left[\frac{\Gamma(n+1)}{S^{n+1}}\right] = t^n \quad (15)$$

Equation (13) in the preliminary (k) can be easily clarified using the integral transform technique, along with equation (3) in the preliminary (b), as demonstrated in[20] ,[12]

4. FRACTIONAL DIFFERENTIAL EQUATION SOLUTIONS

For this part, we suppose that, for a given value of the parameter S, f(t) suffices for the Laplace transform L[f(t)] to converge.

Theorem 4.1: Consider, $1<\lambda<2$ and a_1, $a_2 \in \mathbb{R}$., the fractional differential equation is therefore as follows:

$$f''(t) + a_{1.}\, f^{(\lambda)}(t) + a_2\, f(t) = 0, \qquad (16)$$

The unique solution is found when f (0) =b0 and f' (0) =b1:

$$f(t) = b_0 \sum_{q=0}^{\infty} \frac{(-a_2)^q\, t^{2.q}}{q!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ(\alpha+q+1)\left(-a_1 t^{2-\lambda}\right)^{\alpha}}{Ʈ[(2-\lambda)\alpha+2.q+1]\alpha!} + b_1 \sum_{q=0}^{\infty} \frac{(-a_2)^q t^{2.q+1}}{q!} \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)(-a_1.t^{2-\lambda}).\alpha}{Ʈ\,[(2-\lambda).\alpha+2q+2]\alpha!} + a_1.b_0 \sum_{q=0}^{\infty} \frac{(-a_2)^q t^{2.q-\lambda+1}}{q!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)\left(-a_1.t^{2-\lambda}\right)^{\alpha}}{Ʈ\,[(2-\lambda)\alpha+2.q-\lambda+3]\alpha!} + a_1.b_1 \sum_{q=0}^{+\infty} \frac{(-a_2)^q t^{2q-\lambda+2}}{q\,!} . \sum_{\alpha=0}^{+\infty} \frac{Ʈ\,(\alpha+q+1).\left(-a_1.t^{2-\lambda}\right)^{\alpha}}{Ʈ\,[(2-\lambda).\alpha+2q-\lambda+4]\alpha!} \qquad (17)$$

Proof, Applying the LT to equation (16) and considering the given initial conditions, we get:

$$\mathcal{L}[f(t)]\left(S^2 + a_1.S^{\lambda} + a_2\right) = b_0.S + c.b_0 S^{\lambda-1} + a_1.b_1 S^{\lambda-2} + b_1 \qquad (18)$$

From equation (18), we find:

$$\mathcal{L}[f(t)] = \frac{b_0.S + b_1 + a_{1.}(b_1 S^{\lambda-1} + b_1 S^{\lambda-2})}{S^2 + a_1 S^{\lambda} + a_2} = b_0 \sum_{q=0}^{+\infty} (-a_2)^q . \sum_{q=0}^{+\infty} \binom{q+\alpha}{\alpha} (-a_1)^{\alpha} S^{(\lambda-2).\alpha-2.q-1} + b_1 \sum_{q=0}^{+\infty} (-a_2)^q . \sum_{q=0}^{+\infty} \binom{q+\alpha}{\alpha} (-a_1)^{\alpha} S^{(\lambda-2).\alpha-2.q-2} + a_1.b_0 \sum_{q=0}^{+\infty} (-a_2)^q . \sum_{q=0}^{+\infty} \binom{q+\alpha}{\alpha} (-a_1)^{.\alpha} S^{(\lambda-2).\alpha-2.q+(\lambda-3)} + a_1.b_1 \sum_{q=0}^{+\infty} (-a_2)^q . \sum_{q=0}^{+\infty} \binom{q+\alpha}{\alpha} (-a_1)^{\alpha} S^{(\lambda-2)\alpha-2q+(\lambda-4)} \qquad (19)$$

Given that:

$$\frac{\mathbf{1}}{\boldsymbol{S^2 + a_1.S^{\lambda} + a_2}} = \sum_{q=0}^{+\infty} \frac{(-\mathbf{a_2})^{\mathbf{q}} \boldsymbol{S^{2.q-2}}}{(\mathbf{1} + \mathbf{a_1}.\boldsymbol{S^{\lambda-2}})^{\mathbf{q+1}}} = \sum_{q=0}^{\infty} (-\mathbf{a_{2.}})^{\mathbf{q}} \sum_{\alpha=0}^{\infty} \binom{q+\alpha}{\alpha} (-\mathbf{a_{1.}})^{\boldsymbol{\alpha}} \boldsymbol{S^{(\lambda-2).\alpha-2.q-2}} \qquad (20)$$

Thus, we obtain solution (17) by applying the inverse Laplace transform to equation (19).

$$f(t) = b_0 \sum_{q=0}^{+\infty} \frac{(-a_2)^q t^{2q}}{q\,!} * \sum_{\alpha=0}^{+\infty} \frac{Ʈ\,(\alpha+q+1) * (-a_1 t^{2-\lambda})^{\alpha}}{Ʈ\,[(2-\lambda)\alpha + 2q + 1]\alpha!} + b_1 \sum_{q=0}^{\infty} \frac{(-a_2)^q t^{2q+1}}{q\,!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1) * (-a_1 t^{2-\lambda})^{\alpha}}{Ʈ\,[(2-\lambda)\alpha + 2.q + 2]\alpha!} + a_1.b_0 \sum_{q=0}^{\infty} \frac{(-a_2)^q t^{2.q-\lambda+2}}{q\,!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1) * (-a_1 t^{2-\lambda})^{\alpha}}{Ʈ\,[(2-\lambda)\alpha + 2.q - \lambda + 3]\alpha!} + a_1.b_1 \sum_{q=0}^{\infty} \frac{(-a_2)^q t^{2.q-\lambda+3}}{q\,!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)*(-a_1 t^{2-\lambda})^{\alpha}}{Ʈ\,[(2-\lambda).\alpha+2.q-\lambda+4]\alpha!} \qquad (21)$$

Thus, we observe that the final expression is identical to equation (21).

Example 4.1**:** A fractional differential equation for generalized viscoelastic free damping oscillations.

$$f''(t) + \sqrt{5} f^{\left(\frac{3}{2}\right)}(t) + 10 f(t) = 0 \qquad (22)$$

Considering the starting circumstances f (0) = b0 and f' (0) =b1, and ($a_1 = \sqrt{5}, a_2 = 10$), the equation admits a unique solution, which is:

$$f(t) = b_0 \sum_{q=0}^{\infty} \frac{(-10)^q t^{2q}}{q!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ(\alpha+q+1)\left(-a_1 t^{\left(\frac{3}{2}\right)}\right)^{\alpha}}{Ʈ\left[\left(\left(\frac{1}{2}\right)\alpha+2q+1\right]\alpha!} + b_1 . \sum_{q=0}^{\infty} \frac{(-10)^q t^{2.q+1}}{q\,!} \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)\left(-\sqrt{5} t^{\left(\frac{1}{2}\right)}\right)^{\alpha}}{Ʈ\left[\left(\frac{1}{2}\right)\alpha+2q+2\right]\alpha!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)\left(-\sqrt{5} t^{\left(\frac{1}{2}\right)}\right)^{\alpha}}{Ʈ\left[\left(\frac{1}{2}\right)\alpha+2q+2\right]\alpha!} \sqrt{5}.b_0 \sum_{q=0}^{\infty} \frac{(-10)^q t^{\left(2.q+\frac{1}{2}\right)}}{q\,!} \sum_{\alpha=0}^{\infty} \frac{Ʈ(\alpha+q+1)\left(-\sqrt{5} t^{\left(\frac{1}{2}\right)}\right)^{\alpha}}{Ʈ\left[\left(\frac{1}{2}\right)\alpha+2q+\left(\frac{3}{2}\right)\right]\alpha!} + \sqrt{5}.b_{1.} \sum_{q=0}^{\infty} \frac{(-10)^q t^{\left(2.q+\frac{3}{2}\right)}}{q\,!} * \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1)\left(-\sqrt{5} t^{\left(\frac{1}{2}\right)}\right)^{\alpha}}{Ʈ\left[\left(\frac{1}{2}\right)\alpha+2q+\left(\frac{5}{2}\right)\right]\alpha!} \qquad (23)$$

Upon close examination of Figure 1, the effect of the fractional order λ is evident in accelerating damping and reducing oscillations, as observed in electrical systems such as fractional RLC circuits, and in viscoelastic materials like polymers and biological tissues. when b0= b1=1.

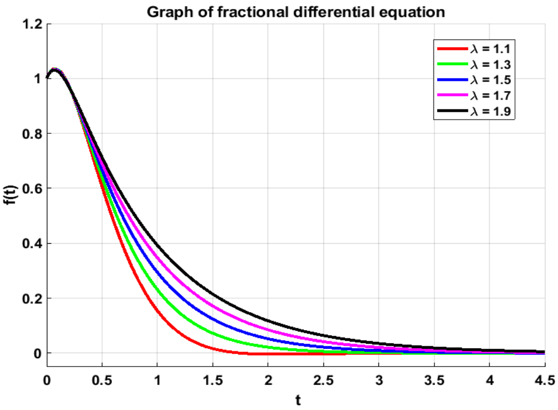

Fig. 1. dynamics of equation (21) resolution

Theorem 4.2, suppose ($1<\lambda<2$), a_1, $a_2 \in \mathbb{R}$.. In this case, the fractional differential equation

$$f^{(\lambda)}(t) + a_1.f'(t) + a_2.f(t) = 0 \qquad (24)$$

The unique solution is found when f (0) =b0 and f' (0) =b1:

$$f(t) = b_0 \sum_{q=0}^{\infty} \frac{Dt^{2.q}}{q\,!} . \sum_{\alpha=0}^{\infty} \frac{Ʈ\,(\alpha+q+1).(-a_1)^{\alpha} t^{A}}{Ʈ\,[(\lambda-1).\alpha+\lambda\, q+1]\alpha!}$$

$$+b_1\sum_{q=0}^{\infty}\frac{D}{q!}\sum_{\alpha=0}^{\infty}\frac{\Gamma(1+\alpha+q)(-a_1)^{\alpha}t^{A+1}}{\Gamma[(\alpha\lambda-\alpha)+\lambda q+2]\alpha!}+$$
$$a_1.b_0\sum_{q=0}^{\infty}\frac{D}{q!}.\sum_{\alpha=0}^{\infty}\frac{\Gamma(1+\alpha+q).(-a_1)^{\alpha}t^{A+\lambda-1}}{\Gamma[(\alpha\lambda-\alpha)+\lambda.q+\lambda].\alpha!} \quad (25)$$

Proof, Applying LT (Preliminary k) and relevant factors, we obtain:

$$S^{\lambda}\mathcal{L}[f(t)]-S^{\lambda-1}.f(0)-S^{\lambda-2}.f'(0)$$
$$+a_1.S\mathcal{L}[f(t)]-a_1.f'(0)+a_2.\mathcal{L}[f(t)]=0$$
$$\Rightarrow(S^{\lambda}+a_1S+a_2)\mathcal{L}[f]=$$
$$b_0.S^{\lambda-1}+b_1.S^{\lambda-2}+a_1.b_1,$$
$$\mathcal{L}[f(t)]=\frac{b_0.S^{\lambda-1}+b_1.S^{\lambda-2}+a_1.b_1}{S^{\lambda}+a_1.S+a_2}=$$
$$b_0\sum_{q=0}^{\infty}(-a_2)^q.\sum_{q=0}^{\infty}\binom{q+\alpha}{\alpha}(-a_1)^{\alpha}S^{A-1}$$
$$+b_1\sum_{q=0}^{\infty}(-a_2)^q.\sum_{q=0}^{\infty}\binom{q+\alpha}{\alpha}(-a_1)^{\alpha}S^{A-2}$$
$$+a_1.b_0\sum_{q=0}^{\infty}(-a_2)^q.\sum_{q=0}^{\infty}\binom{q+\alpha}{\alpha}(-a_1)^{\alpha}S^{A-\lambda} \quad (26)$$

Where, $A=(1-\lambda).\alpha-\lambda.q$

Given that:

$$\frac{1}{S^{\lambda}+a_1.S+a_2}=\frac{S^{-1}}{(S^{\lambda-1}+a_1+a_2.S^{-1})}$$
$$=\frac{S^{-1}}{\left(S^{\lambda-1}+a_1\right).\left(1+\frac{a_2.S^{-1}}{S^{\lambda-1}+a_1}\right)}=$$
$$\sum_{q=0}^{\infty}(-a_2)^q.\sum_{\alpha=0}^{\infty}\binom{q+\alpha}{\alpha}(-a_1)^{\alpha}S^{A-\lambda} \quad (27)$$

From Equation (26), the solution to Equation (25) is obtained via the inverse LT

$$f(t)=$$
$$=b_0\sum_{q=0}^{\infty}\frac{(-a_2)^q t^{2q}}{q!}.\sum_{\alpha=0}^{\infty}\frac{\Gamma(B)(-a_1)^{\alpha}}{\Gamma[C+1]}.\frac{t^A}{\alpha!}$$
$$+b_1\sum_{q=0}^{\infty}\frac{(-a_2)^q}{q!}\sum_{\alpha=0}^{\infty}\frac{\Gamma(B).(-a_1)^{\alpha}}{\Gamma[C+2]}.\frac{t^{A+1}}{\alpha!}$$
$$+a_1b_0\sum_{q=0}^{\infty}\frac{(-a_2)^q}{q!}\sum_{\alpha=0}^{\infty}\frac{\Gamma(B).(-a_1)^{\alpha}}{\Gamma[C+\lambda]}\frac{t^{A+\lambda-1}}{\alpha!} \quad (28)$$

Where, $B=(1+\alpha+q)$, $C=(\lambda-1)\alpha+\lambda q$

The Wright function expresses this solution as,

$$f(t)=b_0\sum_{q=0}^{\infty}\frac{t^{\lambda q}}{q!}.{}_1\Psi_1\left[\left.\begin{matrix}(q+1,1)\\(\lambda.q+1,\lambda-1)\end{matrix}\right|-D\right]+$$
$$b_1\sum_{q=0}^{\infty}\frac{t^{\lambda q+1}}{q!}.{}_1\Psi_1\left[\left.\begin{matrix}(q+1,1)\\(\lambda.q+2,\lambda-1)\end{matrix}\right|-D\right]+$$
$$a_1.b_0\sum_{q=0}^{\infty}\frac{t^{\lambda.q+\lambda-1}}{q!}.{}_1\Psi_1\left[\left.\begin{matrix}(q+1,1)\\(\lambda.q+\lambda,\lambda-1)\end{matrix}\right|-D\right] \quad (29)$$

Where, $D=a_1t^{(\lambda-1)}$, $X=(-a_2)^q$

Example 4.2, Setting $\lambda=3/2$, $a_1=-1$ and $a_2=-2$ in Theorem 4.2, the equation becomes,

$$f^{\left(\frac{3}{2}\right)}(t)=f'(t)+2.f(t), \quad (30)$$

The form of the solution to the previous equation is

$$f(t)=$$
$$b_0.\sum_{q=0}^{+\infty}\frac{(2)^q}{q!}.\sum_{\alpha=0}^{\infty}\frac{\Gamma(1+\alpha+q).(1)^{\alpha}}{\Gamma\left[\left(\frac{1}{2}\right)\alpha+\left(\frac{3}{2}\right)q+1\right]}.\frac{t^{\left(\frac{\alpha}{2}\right)+\left(\frac{3q}{2}\right)}}{\alpha!}$$
$$+b_1\sum_{q=0}^{\infty}\frac{2^q}{q!}\sum_{\alpha=0}^{\infty}\frac{\Gamma(1+\alpha+q)}{\Gamma\left[\left(\frac{\alpha}{2}\right).+\left(\frac{3q}{2}\right)+2\right]}.\frac{t^{\left(\frac{\alpha}{2}\right)+\left(\frac{3q}{2}\right)+1}}{\alpha!}$$
$$-b_0\sum_{q=0}^{\infty}\frac{2^q}{q!}.\sum_{\alpha=0}^{\infty}\frac{\Gamma(1+\alpha+q)}{\Gamma\left[\frac{1}{2}.(\alpha+3q+3)\right]}.\frac{t^{\left(\frac{\alpha}{2}\right)+\left(\frac{3q}{2}\right)+\left(\frac{1}{2}\right)}}{\alpha!}$$

Figure 2 shows how the solution of the fractional differential equation responds for different values of λ when b0= b1=1.

As we can see from the figure as λ increases, the damping becomes stronger. As an applied example of this case Heat diffusion in composite materials, where different components lead to memory-dependent response.

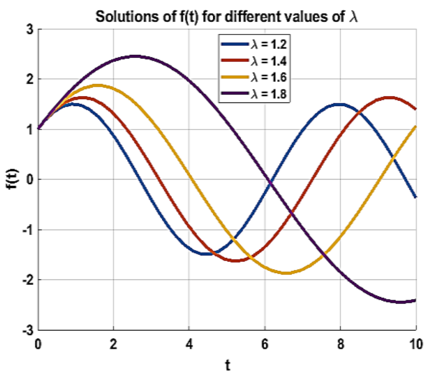

Fig. 2. Solution of f(t) for different values of λ (Example 4.2)

Theorem 4.3, Suppose 0 < λ < 1 and a ∈ R. Then the equation is

$$f^{(\lambda)}(t)-af(t)=0, \quad (31)$$

Considering the starting circumstances y (0) = b0, the solution to equation (31) takes the form:

$$f(t)=b_0.\sum_{q=0}^{\infty}\frac{(a.t^{\lambda})^q}{\Gamma[\lambda.q+1]}=b_0.E_{\lambda,1}(a.t^{\lambda}) \quad (32)$$

Proof, via using the Laplace transform to equation (31), we obtain:

$$S^{\lambda}\mathcal{L}[f(t)]-b_0.S^{\lambda-1}-a.\mathcal{L}[f(t)]=0, \quad (33)$$

Hence, we have,

$$\mathcal{L}[f(t)]=\frac{b_0.S^{\lambda-1}}{b_0.S^{\lambda}-a}=\frac{b_0.S^{-1}}{1-a.S^{-\lambda}}=$$
$$b_0.S^{-1}\sum_{q=0}^{\infty}(a.S^{-\lambda})^q=b_0.\sum_{q=0}^{\infty}a^q.S^{-\lambda.q-1}$$
$$f(t)=b_0.\sum_{q=0}^{\infty}\frac{(a.t^{\lambda})^q}{\Gamma[\lambda.q+1]}=b_0.E_{\lambda,1}(a.t^{\lambda})$$

Remark 4.1, If a1 = 0 in Equation (33), then the equation:

$$f^{(\lambda)}(t) + a_2 f(t) = 0, \text{and } 0 < \lambda \le 2 \quad (33)$$

With the initial conditions f (0) = b0 and $f'(0) = b_1$ has its solution given by

$$f(t) = b_0 . \sum_{q=0}^{+\infty} \frac{(-a_2 . t^{\lambda})^q}{\Gamma\ [1+\lambda q]} + b_1 . t \sum_{q=0}^{+\infty} \frac{(-a_2 . t^{\lambda})^q}{\Gamma\ [2+\lambda q]}, \quad (34)$$

Figure 3 illustrates the plot of solutions of equation (33).

Where we notice, the system stabilizes more quickly as λ increases. As an applied example of this case Drug absorption in biological tissues, where absorption slows due to the memory properties of the medium.

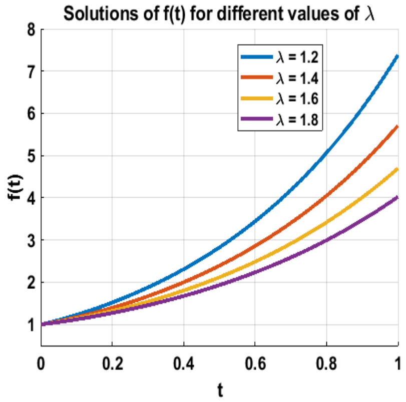

Fig. 3. The specific solution of f(t) for various λ values (Example 4.3).

Theorem 4.4, an equation of nearly simple harmonic oscillation [21], [22]

$$f^{(\lambda)}(t) + W^2 f(t) = 0, \quad 0 < \lambda \le 2 \quad (35)$$

With the initial conditions f (0) = b_0 and $f'(0) = b_1$ has its solution given by

$$f(t) = b_0 E_{\lambda,1}(-W^2 t^{\lambda}) + b_1 t E_{\lambda,2}(-W^2 t^{\lambda}) \quad (36)$$

The proof can be completed by substituting $a_2 = W^2$ into Equation (33)

Figure 4 shows the solution of the nearly simple harmonic oscillation equation (35) for different values of λ. The figure illustrates how varying λ affects the oscillation behavior over time. As an applied example of this case Vibrations of a flexible robotic arm using smart materials like piezoelectrics, which exhibit fractional behavior under stimuli.

5. CONCLUSION

This work illustrates the application of the Laplace transform to the solution of fractional differential equations, highlighting the usefulness of this intricate field. Deeper links are revealed among the LT and other transforms, opening the door to the discovery of further interactions specific to the Laplace transform. A unique approach is presented that uses binomial series extension coefficients in conjunction with the Laplace transform to offer a strong framework for solving these problems. In addition, the research looks at a number of characteristics and provides examples to show how this method might help us understand viscoelastic electrical systems better. By investigating more Laplace transform applications in more intricate electrical engineering contexts, future study can expand on these discoveries.

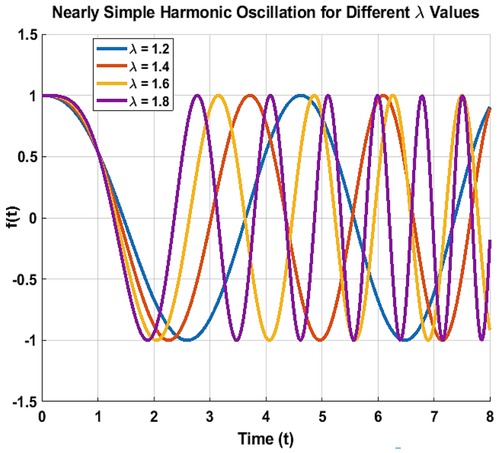

Fig. 4. Nearly Simple Harmonic Oscillation for varying λ

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