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# ANALYSIS OF TRANSVERSE OSCILLATIONS OF DRILLING RIG DERRICK AS TIMOSHENKO BEAM WITH VARIABLE PARAMETERS ALONG THE LENGTH

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Abstract

Free and forced transverse oscillations of a drilling rig tower are considered. The computational model is represented as a Timoshenko beam with variable bending stiffness, running mass, and longitudinal force along the length. It is assumed that the tower is mounted on a rigid platform supported by an elastic base. Additionally, the tower is connected to the base by means of elastic braces. The crown block and rig service platforms attached to the tower are treated as rigid bodies. For the case of harmonic oscillations of a Timoshenko beam with variable parameters along its length, the differential equations of the amplitude functions are obtained and reduced to Volterra integral equations. Oscillations of a multi-span structure are calculated using the matrix method of initial parameters. An analysis of the results of calculations of transverse oscillations of a drill tower is presented.

Keywords: drilling rig, derrick, Timoshenko beam with variable characteristics, transverse oscillations

## 1. INTRODUCTION

Modern drilling rigs are complex sets of equipment and structures that differ significantly in both design and technical characteristics. By functional purpose, the following main drilling rig systems can be distinguished: lifting, which is used to lower a pipe string into the well and raise it; rotary (rotor system), which drives the actuator in the process of deepening the well; circulation, which is used to flush the well from drilled rock by forcing the solution to circulate. Drilling towers or masts are used to install hoist mechanisms, devices for mechanizing downhole operations and placing drill spark plugs.

Tower drilling towers and A-shaped drilling masts are widely used. Tower towers have relatively high rigidity and high strength. These structures are usually made in the form of a tetrahedral pyramid, the edges of which are made of pipes or rolled sections and connected to each other in the planes of the edges by rod elements. Mast-type structures are characterized by reduced metal consumption, good transportability and relative ease of installation. They consist of two composite rods made in the form of spatial trusses, which are interconnected in several places by transverse beams. The A-shaped masts are mounted on support hinges and additionally secured in a vertical position with struts. For drilling wells to a depth of up to 3000 m, A-shaped masts are mainly used; if the drilling depth is 3000-5000 m, both masts and towers are used; for drilling to a depth of more than 5000 m, tower drilling towers are used.

When performing dynamic calculations, it is advisable to consider the legs of drilling masts and steel structures of towers as solid bars (composite rods). The transverse oscillations of composite rods can be described with sufficient accuracy for practice by partial differential equations that take into account, according to the Timoshenko beam theory, bending and shear deformations and inertia of translational and rotational motion of elementary segments of the composite rod.

The peculiarity of calculating a tower drilling tower for transverse oscillations is that it has a variable bending stiffness, distributed mass, and longitudinal force along its length. In addition, it is necessary to take into account the additional fastening of the structure to the ground by means of braces, as well as the presence of structures connected to the tower in the form of pronounced solids, for example, platforms for servicing the drilling rig, crown block, etc. All of this greatly complicates the study of dynamic phenomena that occur during the operation of drilling towers.

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Paper [1] briefly presents a variational formulation of the theory of spatial nonlinear classical beams of constant cross-section, followed by consideration of the plane case. The term "classical beams" here refers to generalized one-dimensional continuous media that model the mechanical behavior of three-dimensional beam objects. If the one-dimensional continuum corresponds to a deformable curve in space parameterized by a single material coordinate and time, then the generalized continuum is supplemented by additional kinematic quantities that depend on the same parameters. The authors introduce three nonlinear spatial beams: the Timoshenko beam, the Euler-Bernoulli beam, and the inextensible Euler-Bernoulli beam. In the spatial theory, the Euler-Bernoulli beam and its inextensible variant are considered as bounded theories. In the plane case, both constrained theories are further described by means of alternative kinematics that satisfy the basic constraints of these theories.

In [2], the governing equations of oscillation of a beam with a moderately large deflection and an arbitrary cross section were obtained using the first-order theory of shear deformation. The beam is homogeneous, isotropic, and subjected to axial loads. The kinematics of the problem is described by the von-Karman displacement dependence of deformations, and the governing equations are based on Hooke's law. The system of partial differential equations describing the axial and transverse oscillations of homogeneous beams contains four coupled nonlinear equations with variable coefficients obtained by applying the Hamilton's principle. The Galerkin method in combination with the perturbation method was used to determine the linear natural frequencies. To confirm the validity of the oscillation analysis, the analytical results are compared with the corresponding numerical results obtained by finite element analysis.

Based on Timoshenko's theory of beams, a model of a transverse asymmetric cantilever beam was developed in [3], taking into account its shear deformations and special boundary conditions. The analytical and numerical solution of the problem in the form of the amplitude function and the circular frequency of oscillations of the cantilever beam was obtained using the method of transfer matrices and the Newton-Raphson iterative method. The accuracy and efficiency of the method are verified by comparing theoretical results with experimental data and known analytical and numerical solutions.

In [4], free transverse oscillations of a rod of constant cross-section with regard to its own weight are considered. The corresponding partial differential equation of transverse oscillations of the rod is reduced to two ordinary differential equations with respect to the time function and the amplitude function of deflections. As a result, an exact solution of the original differential equation of transverse oscillations of the rod expressed in dimensionless fundamental functions and initial parameters is obtained. The proposed approach makes it possible to study free oscillations of a rod with different types of boundary conditions.

To establish the relationship between sound generation and oscillation of the walls of composite pipes of variable cross-section, a set of composite cones made of different materials with walls of different thicknesses was manufactured, theoretically studied, and experimentally tested [5]. The main purpose of the experimental studies was to develop such composite cones, excite standing waves in them at certain frequencies, and record acoustic radiation and corresponding wall oscillations for each of them. A variation of the Webster's wave equation was used as the basis for theoretical studies.

Paper [6] considers a one-dimensional finite element model of the nonlinear dynamic behavior of thin-walled composite beams with an open variable cross-section under an arbitrary external dynamic load. The model takes into account torsion or bending with torsion without any assumptions about the deformation amplitudes. The algorithm for solving the problem is a special case of an efficient family of high-order implicit algorithms based on time discretization of unknown functions, introduction of an artificial parameter, and use of an asymptotic numerical method.

Study [7] proposes a complex model of a mechanical system formed by a Timoshenko beam and a Vlasov base with arbitrary boundary conditions provided by the introduction of restraining springs in the translational and rotational directions. The solutions to the problem in the form of beam responses to arbitrarily distributed loads are strictly derived taking into account spatial changes in soil parameters. The reliability of the theoretical results is confirmed by experimental data and by comparing the analytical closed-form solutions with the results of numerical modeling. The results show that the proposed analytical model can be applied not only to homogeneous reservoirs and common boundary conditions, but also to heterogeneous reservoirs and arbitrary boundary conditions. The Vlasov-Timoshenko model is quite versatile, as it can be degenerated into the Vlasov-Euler-Bernoulli model, the Winkler-Timoshenko model, and the Winkler-Euler-Bernoulli model. The proposed model is applied to evaluate the longitudinal response of a panel tunnel caused by adjacent excavations.

Paper [8] investigated the forced longitudinal oscillations of a fractional-type rheological rod with a variable cross section. A fractional partial differential equation and its individual solutions for forced longitudinal oscillations of the rod were obtained. For the main cases of boundary conditions and different forms of oscillations, the eigen amplitude functions are obtained. The Laplace transform and the properties of the three functions under convolution are used to solve the fractional order differential equations. The energy analysis of the fractional-order system is carried out. The rate of dissipation of mechanical energy of longitudinal oscillations of the rod is determined.

The study [9] consists in analyzing the free oscillations of beams consisting of functionally graded porous materials and characterized by a variable cross-section along the length. A wide range of beams is studied, including conical configurations, various profiles, and porosity characteristics of beam materials. The equations of motion are derived on the basis of Hamilton's principle within the framework of Timoshenko's beam theory. They are solved using differential transformations by an analytical and numerical method that has been adapted to take into account various boundary conditions. To validate the proposed solution technique, the calculated eigenfrequencies are compared with the available literature results for conical inhomogeneous beams and homogeneous porous beams. New results are obtained for conical porous beams with a variable cross section.

Article [10] is devoted to the study of oscillations of a nonlinear beam under the action of a moving load, and the dissipation of mechanical energy of the system is described by means of the fractional Caputo derivative. The nonlinearities arise from the assumption of moderately large displacements of the beam. In accordance with the procedure for solving the problem using the Galerkin method, deflections are presented in the form of linear waveforms and generalized time-dependent coordinates. Numerical results are presented to evaluate the accuracy of the proposed calculation method, as well as to study the influence of the order and coefficient of the fractional derivative on nonlinear oscillations of the beam.

Paper [11] investigates the nonlinear deflection of an infinite beam with a variable cross-section, mounted on a nonlinear elastic base. The authors apply a new analytical and numerical method of pseudo-parametric iteration, suitable for solving both ordinary differential equations and partial differential equations. Six types of infinite beams under static loading are considered. The Bernoulli-Euler beam equation is used to calculate the nonlinear deflection of an infinite beam with a variable cross section. The high efficiency of the method is illustrated.

An effective approach to obtaining exact solutions to the problem of free oscillations of functionally gradient beams with variable cross-sections resting on a Pasternak elastic base is considered in [12]. Using the method of separation of variables and the Laplace transform based on two-dimensional elasticity theory, general expressions of displacements and stresses are obtained that fully follow from the dynamic differential equations and boundary conditions for functionally gradient beams with arbitrarily variable cross-sections. The frequency equations of free oscillation of the beams are obtained by expanding the solutions into a Fourier series with consideration of the boundary conditions. The validity of the developed approach, as well as its efficiency and accuracy, are confirmed by analyzing several typical examples.

Paper [13] investigates the dynamic responses of a beam whose cross-sectional area and moment of inertia are varied by power functions. The eigenfrequencies and oscillation modes of the beam were calculated using a semi-analytical method. To verify the accuracy of the calculation, the analytical solutions are compared with those found by the finite element method. Taking into account the convective inertial force, relative inertial force, Coriolis force, and centrifugal inertial force, the equation of transverse bending oscillations of a beam under the action of a moving body is obtained and solved by the Newmark's numerical method. The dynamic response of a cantilevered beam with variable cross-section and moment of inertia is analyzed by an example.

Paper [14] considers a nonlinear mathematical model of transverse oscillations of an aboveground pipeline section during the movement of a diagnostic device. The dynamic analysis is performed by the method of generalized displacements. The pipeline is considered as a link with distributed parameters and the diagnostic device as a solid body. The equations of motion of the mechanical system are derived using Lagrange's equations of the second kind. The influence of mechanical system parameters and device speed on pipeline deflections and bending moments in the pipe is illustrated.

A large number of rails with a curved axis and variable cross-section are used in railway switches, which necessitates modeling the stress-strain state and oscillations of curved rods. Paper [15] presents a general formulation of the problem of analyzing free and forced oscillations of a curved Timoshenko beam of variable cross section and its implementation in application to the dynamic interaction of a train and a turnout. First, by finding the eigenvalues, the eigenfrequencies and shapes of the beam oscillations are calculated. Then, the solution to the problem of forced oscillations is obtained using modal superposition and orthogonality. A comparative analysis of the results obtained with those obtained using the finite element method is carried out.

Paper [16] investigated the influence of nonlinear relations on the behavior of the electromechanical coupling in inhomogeneous piezoelectric semiconductor rods. Based on the three-dimensional theory of a piezoelectric semiconductor and the double energy series, a one-dimensional bending model was developed. An iterative procedure based on the differential-quadrature method is proposed to solve the nonlinear problem. The calculation results are compared with the results of the finite element method. It is shown that the proposed method has good convergence and high accuracy. The effects of nonlinearity and inhomogeneity in a piezoelectric semiconductor are thoroughly discussed in the paper.

Paper [17] analyzes the transverse oscillations of a transmission line pylon with a variable cross-section along its length. Such structures are subjected to the simultaneous effects of internal and external loads, which leads to the occurrence of transverse and longitudinal oscillations. These oscillations are described by two partial differential equations that include two unknown functions. If the oscillations are small, then the terms that connect the equations of transverse and longitudinal oscillations can be neglected, considering each type of oscillation separately.

The drill tower is an important element of the drilling rig that ensures the reliability of the drilling complex during the construction of both exploration and production wells. Particularly dangerous is the loss of fatigue strength of parts and assemblies of load-bearing structures due to their intense oscillations. In [18], a finite element model of the JJ60/38-W drilling tower was developed and its oscillations were investigated, taking into account axial, bending, and torsional deformations. Free and harmonic forced oscillations of the drill tower components are investigated. The ANSYS software was used in the work, and the results obtained are of practical interest for studying real processes of well deepening.

Paper [19] notes that the hook load, the dead weight of the rig elements, and the wind load have a decisive influence on the strength and reliability of drilling towers in offshore conditions. During the design of drilling towers, their static calculations for strength and reliability are mostly carried out, taking into account the laws of load distribution as random variables, using the methods of mathematical statistics. In this paper, a spatial model of a typical drilling tower is built using APDL (ANSYS Parametric Design Language), and its reliability is investigated using Python and APDL code. The influence of each load on the strength and reliability of the structure is determined. This study provides a thorough explanation of the law of distribution of strength and reliability of drilling towers under complex loads.

Paper [20] considers unsteady processes in the lifting system of a drilling rig and their impact on the durability of structural elements. The mechanical system includes links with both concentrated and distributed parameters. The drillpipe string is considered as a stepped rod with longitudinal oscillations. The nonlinear mathematical model of dynamic phenomena takes into account the interconnection of electromagnetic processes in induction motors with mechanical oscillations. The durability of structural elements is estimated using the NASGRO equation to determine the crack growth rate.

Paper [21] proposes a method for assessing the fatigue strength of steel truss suspension bridges under unsteady and non-Gaussian loading under the action of mountain wind. The normal wind speed was obtained by synthesizing harmonic waves with consideration of their nonstationary characteristics. Using the Palmgren-Miner rule, the influence of mechanical system reactions on fatigue damage of suspension bridges is studied. The research results show that fatigue damage to the main cable of a suspension bridge is greatest in the middle part of the span. Under the influence of unsteady wind flow, the level of damage to the elements of the truss wind structure increases.

Paper [22] investigates the dynamic behavior of a periodic structure consisting of rods with a variable cross-sectional area. The cells of the structure have two rods each. In one case, the cross-sectional area of the rods varies exponentially, and in the other, linearly. The results of numerical modeling of dynamic processes are in good agreement with the experimental results. However, a peculiarity was found: experimental studies revealed bending oscillations that were not taken into account in the theoretical model. It was found that asymmetric cells perform better and have a wider attenuation band compared to symmetric cells.

Paper [23] is devoted to the study of the elastic stability of double conical microbeams embedded in a Winkler-type elastic base. It is assumed that the cross-section of the beams varies in the longitudinal direction according to a linear law. The theory of nonlocal stress pairs and the Bernoulli-Euler theory are used to obtain the beam size-dependent stress-strain equation. The minimum eigenvalue of the characteristic equation and, accordingly, the critical load of a conical microbeam are determined by the Rayleigh-Ritz method. The influence of the taper coefficient, nonlocal parameters, microbeam length scale parameters, and foundation characteristics on the stability of the elastic system is comprehensively investigated. Prefabricated prestressed frame beams with variable cross-section equipped with anchor cables are used as new support systems for slope reinforcement. The design and construction of such systems encounters certain difficulties due to the complex nature of the distribution of internal forces between system elements, as well as within elements with variable characteristics.

Paper [24] proposes a method for calculating internal forces and displacements of prestressed beams with a variable cross section mounted on a Pasternak elastic base. The calculation was performed using the finite difference theory. The bending moments obtained by the analytical method, field tests, and numerical modeling are in general agreement, which proves the feasibility of practical application of the obtained calculation method.

In [25], using the variational principle and the method of transition matrices, analytical and numerical solutions were obtained for the displacements and internal forces of a Timoshenko beam of variable cross section mounted on a Pasternak elastic base. The obtained solution is compared with the results of finite-difference analysis, which confirms the accuracy and reliability of the proposed theory. The results of the theoretical calculations are in good agreement with the results of monitoring the technical facility under construction. The model degenerates into the Winkler-Timoshenko model when the stiffness of the shear layer of the foundation tends to zero.

Paper [26] investigates the nonlinear dynamic behavior of a functionally graded Euler-Bernoulli beam supported on a fractional viscoelastic Pasternak base subjected to harmonic loads. The von Karman strain-displacement relation is used to describe the nonlinear geometric behavior of the beam. To represent the deformation of the material in the thickness direction of a functionally graded beam, a step model is used. First, the basic equation of motion is derived using Hamilton's principle and then reduced to a nonlinear differential equation of fractional order using the single-mode Galerkin approximation. The methodology for determining the stationary amplitude-frequency characteristics using the harmonic balance method and the continuation technique is presented.

The analysis of free and steady-state forced oscillations of drilling towers can be performed using the method of initial parameters. In this case, the most difficult task is to construct a transition matrix for a long-dimensional structure with variable parameters along its length, which is associated with the integration of the differential equation of amplitude functions with variable coefficients. This article is devoted to solving this equation, forming a transition matrix, and applying the results to determine the frequencies and shapes of free oscillations and amplitudes of forced harmonic oscillations of a drill tower. The proposed method of mathematical modeling of long-dimensional metal structures can also be used in studies of the dynamics of drill masts

## 2. EQUATION OF TRANSVERSE OSCILLATIONS OF THE DRILLING TOWER AND EQUATIONS OF AMPLITUDE FUNCTIONS

The design scheme of a high-rise building is shown in Fig. 1. The drill tower consists of *n-1* spans with lengths equal to $l_1, l_2, ..., l_{n-1},$ respectively. At the boundaries of the runs, it is assumed that there are rigid elements with masses $m_1, m_2, ..., m_n$ and central moments of inertia $J_1, J_2, ..., J_n$, as well as elastic supports with stiffness coefficients in the horizontal and rotational directions $c_{w1}, c_{w2}, ..., c_{wn}$; $c_{\varphi 1}, c_{\varphi 2}, ..., c_{\varphi n}$, and the coefficients of viscous friction are $\nu_{w1}, \nu_{w2}, ..., \nu_{wn}$; $\nu_{\varphi 1}, \nu_{\varphi 2}, ..., \nu_{\varphi n}$. In the absence of a particular elastic element of the support, the values of its elastic-dissipative parameters should be assumed equal to zero.

The equations of oscillation of a structure follow from the conditions of dynamic equilibrium of an elementary segment of a high-rise structure, which is considered as a S. Timoshenko beam. The derivation

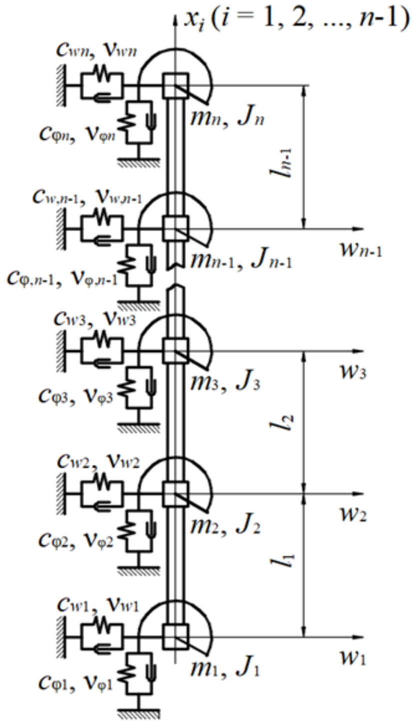


Fig. 1. Design schema of the drilling tower

of such equations for rods of constant cross-section, taking into account the action of axial forces, is given in [27, 28]. A more detailed analysis of the forces acting on an infinitesimal element is given in [29]. In accordance with this methodology, we develop the equations of transverse oscillations of the structure, taking into account that the moment of inertia of the tower cross section, its distributed mass, and axial force are continuous functions of the longitudinal coordinate,

$$\frac{\partial}{\partial x_i}\left(EI_i\frac{\partial \varphi_i}{\partial x_i}\right)+\kappa_i GA_i\left(\frac{\partial w_i}{\partial x_i}-\varphi_i\right)-I_i\rho_i\frac{\partial^2\varphi_i}{\partial t^2}=0\ ;$$

$$\rho_i A_i\frac{\partial^2 w_i}{\partial t^2}-\kappa_i GA_i\left(\frac{\partial^2 w_i}{\partial x_i^2}-\frac{\partial \varphi_i}{\partial x_i}\right)+N_i\frac{\partial^2 w_i}{\partial x_i^2}=0$$

$$(i=1,\ 2,\ \dots,\ n-1),\qquad(1)$$

where $E$ and $G$ are the elastic moduli of the first and second kind; $I_i$ and $A_i$ are the axial moment of inertia and the cross-sectional area of the rod span; $\rho_i$ *is* the average density of the material; $\kappa_i$ the coefficient characterizing the effect of shear deformation; $N_i$ is the longitudinal force; $w_i$ the deflection; $\varphi_i$ the angle of inclination of the tangent to the bent axis of the rod under the action of bending moments; $x_i$ is the longitudinal coordinate; $t$ is time.

The integrals of the differential equations (1) are written in the form

$$w_i(x_i,\ t)=W_i(x_i)\,exp(\lambda t)\ ;$$
$$\varphi_i(x_i,\ t)=\Phi_i(x_i)\,exp(\lambda t),\qquad(2)$$

where $W_i(x_i)$ and $\Phi_i(x_i)$ are amplitude functions; $\lambda$ is the eigenvalue. A similar form of writing the unknown functions of S. Tymoshenko's equations was used in [31]. In this case, it is the most convenient and makes it possible to simplify the solution of the problem.

Substituting expressions (2) into equations (1), we obtain

$$\frac{d}{dx_i}\left(EI_i\Phi_i^{'}\right)+\kappa_i GA_i\left(W_i^{'}-\Phi_i\right)-I_i\rho_i\lambda^2\Phi_i=0;$$
$$\rho_i A_i\lambda^2 W_i-\kappa_i GA_i\left(W_i^{''}-\Phi_i^{'}\right)+N_iW_i^{''}=0$$
$$(i=1,\ 2,\ \dots,\ n-1). \qquad (3)$$

Excluding from the last relations one of the unknown functions $\Phi_i$, we write the equations of the amplitude functions as

$$\sum_{j=0}^{4} W_i^{(\mathrm{IV}-j)}\sum_{k=0}^{3}\lambda^{2k} f_{ijk}(x_i)=0, \qquad (4)$$

where

$$f_{i00}=1;\quad f_{i01}=\frac{I_i\rho_i}{s_i};\quad f_{i02}=0;\quad f_{i03}=0;$$
$$f_{i10}=\frac{2I_i^{'}}{I_i}-\frac{2N_i^{'}}{r_i};\quad f_{i11}=\frac{q_i}{s_i}-\frac{2N_i^{'}I_i\rho_i}{s_i^2 r_i};$$
$$f_{i12}=0;\quad f_{i13}=0;$$
$$f_{i20}=\frac{I_i^{''}}{I_i}-\frac{s_i^2}{EI_i}+\frac{s_i^4}{EI_i r_i}-\frac{2I_i^{'}N_i^{'}}{I_i r_i}-\frac{N_i^{''}}{r_i};$$
$$f_{i21}=\frac{\rho_i s_i^2}{Er_i}+\frac{I_i^{''}\rho_i}{s_i^2}-\frac{2\rho_i}{E}-\frac{I_i^{'}\vartheta_i}{I_i s_i^2}-\frac{2I_i^{'}\rho_i N_i^{'}}{r_i s_i^2}+\frac{h_i}{r_i s_i^2};$$
$$f_{i22}=-\frac{I_i\rho_i^2}{Es_i^2}-\frac{I_i\rho_i^2 A_i}{r_i s_i^2};\quad f_{i23}=0;$$
$$f_{i30}=0;\quad f_{i31}=-\frac{\vartheta_i(2EA_i+s_i^2)}{EI_i r_i};$$
$$f_{i32}=-\frac{\vartheta_i\rho_i A_i}{r_i s_i^2};\quad f_{i33}=0;$$
$$f_{i40}=0;\quad f_{i41}=\frac{\rho_i A_i s_i^2-EA_i\vartheta_i^{'}}{EI_i r_i};$$
$$f_{i42}=\frac{2A_i\rho_i^2}{Er_i}+\frac{A_i\vartheta_i^2-A_iI_i\rho_i\vartheta_i^{'}}{I_i r_i s_i^2};\quad f_{i43}=\frac{A_iI_i\rho_i^3}{Er_i s_i^2}.$$

Here:

$$s_i^2=\kappa_i GA_i;\quad r_i=s_i^2-N_i;\quad \vartheta_i=I_i^{'}\rho_i+I_i\rho_i^{'};$$
$$q_i=I_i^{'}\rho_i-I_i\rho_i^{'};\quad h_i=\vartheta_i N_i^{'}-s_i^2\rho_i A_i-I_i\rho_i N_i^{''}.$$

## 3. REDUCTION OF DIFFERENTIAL EQUATIONS OF AMPLITUDE FUNCTIONS TO INTEGRAL EQUATIONS AND THEIR SOLUTION

Taking $u_i(x_i)=W_i^{\mathrm{IV}}(x_i)$, we reduce the differential equation (4) to a normal integral equation:

$$\int_0^{x_i} K_i(x_i,\ \xi,\ \lambda)u_i(\xi)d\xi+u_i(x_i)\sum_{i=0}^{3}\lambda^{2j}f_{i0j}(x_i)=$$
$$=-\sum_{k=0}^{3} W_{i0}^{(k)}\sum_{l=0}^{3}\lambda^{2l}\sum_{j=0}^{k} f_{i,\ 4-j,\ l}(x_i)\frac{x_i^{3-j}}{(3-j)!} \qquad (5)$$

with kernel

$$K_i(x_i,\ \xi,\ \lambda)=\sum_{j=0}^{3}\lambda^{2j}K_{ij}(x_i,\ \xi),$$

where

$$K_{ij}(x_i,\ \xi)=\sum_{k=1}^{4} f_{ikj}(x_i)\frac{(x_i-\xi)^{k-1}}{(k-1)!}.$$

The value *of* $W_{i0}^{(k)}$ in Equation (5) is the derivative of order $k$ of the function $W_i(x_i)$ in the initial ($x_i=0$) cross-section of the run; $\xi$ is the integration variable. The solution of the integral equation (5) is found in the form of a power series with respect to the parameter λ,

$$u_i(\lambda,\ x_i)=u_{i0}(x_i)+\lambda^2u_{i1}(x_i)+\lambda^4u_{i2}(x_i)+\cdots\ . \qquad (6)$$

Substituting expression (6) into equation (5) and equating the coefficients at the same powers of the parameter λ of the left and right sides of the resulting equality, we arrive at the sequence of integral relations

$$u_{ij}(x_i)+\int_0^{x_i}K_{i0}(x_i,\ \xi)u_{ij}(\xi)d\xi+R_{ij}=0 \qquad (7)$$
$$(j=0,\ 1,\ \dots),$$

where

$$R_{i0}=\Theta_{i0};$$
$$R_{ij}=\sum_{k=1}^{j} f_{i0k}(x_i)u_{j-k}(x_i)+$$
$$+\sum_{k=1}^{j}\int_0^{x_i}K_{ik}(x_i,\ \xi)u_{j-k}(\xi)+\Theta_{ij}\quad (j=1,\ 2,\ 3);$$
$$R_{ij}=\sum_{k=1}^{j} f_{i0k}(x_i)u_{j-k}(x_i)+$$
$$+\sum_{k=1}^{j}\int_0^{x_i}K_{ik}(x_i,\ \xi)u_{j-k}(\xi)d\xi\quad (j=4,\ 5,\ \dots).$$

Here

$$\Theta_{il}=\sum_{k=0}^{3}W_{i0}^{(k)}\lambda^{2l}\sum_{j=0}^{k}f_{i,\ 4-j,\ l}(x_i)\frac{x_i^{3-j}}{(3-j)!}$$
$$(l=0,\ 1,\ 2,\ 3). \qquad (8)$$

Relationships (7) are Volterra equations of the second kind, in the process of solving which we determine the functions $u_{i0}$, $u_{i1}$, $u_{i2}$, ... .

To find the responses of the function $u_i(\lambda,\ x_i)$ to unit jumps in $W_{i0}^{(k)}$ ($k=0,\ 1,\ 2,\ 3$), when solving equations (7), only the term of expression (8) containing the derivative of the function $W_i(x_i)$ of the corresponding order at the point with coordinate $x_i=0$ should be taken into account.

Analyzing the process of sequential determination of the functions $u_{i0}$, $u_{i1}$, $u_{i2}$, ... using equations (7), we can see that the series (6) does not always converge. For certain values of the functions $f_{ijk}$ ($j=1,\ 2,\ 3,\ 4$; $k=0,\ 1,\ 2,\ 3$) and a sufficiently large value of the modulus of the parameter λ, the terms of this series increase with the increase of the ordinal number.

Let us determine the conditions under which the infinite sum (6) has a limit. Suppose that for $0\le x_i\le l_i$ there exist the following limits on the absolute values of the functions:

$$|f_{i01}|\le\alpha_i;\quad |K_{ij}|\le\beta_{ij};\quad |u_{ij}|\le\gamma_{ij};$$
$$|f_{i,\ k,\ j+1}^{*}|\le\delta_{i,\ k,\ j+1}\quad (j=0,\ 1,\ 2,\ 3). \qquad (9)$$

Here

$$f_{i,\ k,\ j+1}^{*}=\sum_{l=0}^{k}f_{i,\ 4-l,\ j-2}(x_i)\frac{x_i^{3-l}}{(3-l)!}$$
$$(k=0,\ 1,\ 2,\ 3).$$

Applying equation (7) and ratio (9), we obtain:

$$\gamma_{i0}\le\zeta_i\delta_{ik1};$$
$$\gamma_{i1}\le\zeta_i[\gamma_{i0}(\alpha_i+\beta_{i1}l_i)+\delta_{ik2}];$$
$$\gamma_{i2}\le\zeta_i[\gamma_{i1}(\alpha_i+\beta_{i1}l_i)+\gamma_{i0}\beta_{i2}l_i+\delta_{ik3}];$$

$$\gamma_{i3} \leq \zeta_i[\gamma_{i2}(\alpha_i + \beta_{i1}l_i) + \gamma_{i1}\beta_{i2}l_i + \gamma_{i0}\beta_{i3}l_i + \delta_{ik4}];$$
$$\gamma_{in} \leq \zeta_i[\gamma_{i,\,n-1}(\alpha_i + \beta_{i1}l_i) + \gamma_{i,\,n-2}\beta_{i2}l_i + \gamma_{i,\,n-3}\beta_{i3}l_i]$$
$$(n = 4,\ 5,\ \dots),$$
where
$$\zeta_i = \frac{1}{|1-\beta_{i0}l_i|}.$$

Consider the auxiliary number series
$$s_0 + \lambda^2 s_1 + \lambda^4 s_2 + \lambda^6 s_3 + \cdots \quad , \tag{10}$$
whose members are defined by the relations
$$s_{i0} = \zeta_i\delta_{ik1};$$
$$s_{i1} = \zeta_i[s_{i0}(\alpha_i + \beta_{i1}l_i) + \delta_{ik2}];$$
$$s_{i2} = \zeta_i[s_{i1}(\alpha_i + \beta_{i1}l_i) + s_{i0}\beta_{i2} + \delta_{ik3}];$$
$$s_{i3} = \zeta_i[s_{i2}(\alpha_i + \beta_{i1}l_i) + s_{i1}\beta_{i2} + s_{i0}\beta_{i3} + \delta_{ik4}];$$
$$s_{in} = \zeta_i[s_{i,\,n-1}(\alpha_i + \beta_{i1}l_i) + s_{i,\,n-2}\beta_{i2} + s_{i,\,n-3}\beta_{i3}]$$
$$(n = 4,\ 5,\ \dots).$$

As can be seen from the last expressions, if the following conditions are met
$$\delta_{ik1} > \frac{|\lambda^2|}{1-\zeta_i\alpha_i|\lambda^2|}(\zeta_i\beta_{i1}\delta_{ik1}l_i + \delta_{ik2});$$
$$\delta_{ik2} > (l_i\gamma_{i0}\beta_{i2} + \delta_{ik3})|\lambda^2|;$$
$$\delta_{ik3} > (\gamma_{i0}\beta_{i3} + \delta_{ik4})|\lambda^2| \tag{11}$$
inequality is satisfied
$$s_{n+1}\lambda^2 < s_n \ \ (n = 1,\ 2,\ \dots).$$

Taking into account the third criterion of series convergence and d'Alembert's criterion, the condition for the convergence of series (10) is given as
$$\zeta_i[(\alpha_i + l_i\beta_{i1}) + l_i\beta_{i2}\lambda^2 + l_i\beta_{i3}\lambda^4] \cdot |\lambda^2| < 1. \tag{12}$$

Since there is a correlation between
$$|u_{ij}| \leq \gamma_{ij} \leq s_{ij} \quad (j = 0,\ 1,\ 2,\ 3,\ \dots),$$
then, in the case of convergence of series (10), series (6) is major for any value of the parameter λ that satisfies relations (11), (12).

The need to meet these conditions limits the value of the oscillation frequency at which this calculation method can be used in practice. However, it can be successfully used to analyze low-frequency oscillations of the system, which are usually of the greatest practical interest. When studying free oscillations, relations (11), (12) mostly make it possible to determine several lower natural frequencies and shapes of a mechanical system.

Thus, the function $u_i(\lambda, x_i)$ can be represented as
$$u_i(\lambda, x_i) = W_{i0}u_{i1}^* + W_{i0}'u_{i2}^* + W_{i0}''u_{i3}^* + W_{i0}'''u_{i4}^*, \tag{13}$$
where $u_{ik}^*(\lambda, x_i)$ ($k$ = 1, 2, 3, 4) are solutions of equations (5) obtained with respect to relation (8).

Determining the functions $W_i^k(x_i)$ ($k$ = 0, 1, 2, 3) by successive integration of expression (13) makes it possible to construct the fundamental matrix for equation (4).

## 4. BOUNDARY CONDITIONS. APPLICATION OF THE MATRIX METHOD OF INITIAL PARAMETERS

The amplitude values of the bending moment and transverse force are defined as
$$\bar{M}_i = -EI_i\Phi_i';\bar{Q}_i = \kappa_i GA_i(W_i' - \Phi_i). \tag{14}$$

In accordance with [29], the amplitude value of the horizontal component of the internal force will be
$$\bar{V}_i = \bar{Q}_i - \bar{N}_i W_i'. \tag{15}$$

Applying the fundamental Cauchy system obtained for the differential equation (4), as well as relations (3), (14), (15), we obtain the relationship between displacements and internal forces at the ends of the structure span
$$Y_i(l_i) = F_i^*(l_i)Y_i(0) \quad (i = 1,\ 2,\ \dots,\ n-1), \tag{16}$$
where
$$F_i^*(l_i) = A_{1i}F^{**}(l_i)A_{2i}; \tag{17}$$
$$Y_i(x_i) = \mathrm{col}[W_i(x_i),\ \Phi_i(x_i),\ \bar{M}_i(x_i),\ \bar{V}_i(x_i)]$$
$$(x_i = 0, l_i). \tag{18}$$

Here, $F_i^{(**)}$ is the fundamental matrix; $A_{1i}$ and $A_{2i}$ are the matrices of the connection of $W_i^{(k)}$ ($k$ = 1, 2, 3, 4) with the elements of the column matrix (18).

The square matrices $A_{1i}$ and $A_{2i}$ are defined by the expressions
$$A_{1i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{\Psi_{1i}}{\Omega_i^*} & -\frac{\Psi_{4i}}{\Omega_i^*} & \frac{\Psi_{5i}}{\Omega_i^*} & \frac{EI_ir_i}{\Omega_i^*} \\ \frac{\Psi_{2i}}{s_i^2} & 0 & -\frac{EI_ir_i}{s_i^2} & 0 \\ \frac{\Psi_{1i}}{\Omega_i} & \frac{\Psi_{6i}}{\Omega_i} & -\frac{\Psi_{5i}}{\Omega_i} & -\frac{EI_ir_i}{\Omega_i} \end{bmatrix}; \tag{19}$$

$$A_{2i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s_i^2r_i^{-1} & 0 & r_i^{-1} \\ \rho_iA_ir_i^{-1}\lambda^2 & 0 & -s_i^2\eta_i & 0 \\ \Psi_{7i}r_i^{-2} & \Psi_{8i}r_i^{-1} & \Psi_{9i}I_i^{-1}r_i^{-1} & \Psi_{4i}r_i^{-1}\eta_i \end{bmatrix}, \tag{20}$$
where
$$\Omega_i = s_i^2 + I_i\rho_i\lambda^2; \quad \Omega_{1i}^* = s_i^2\Omega_i; \quad \eta_i = (EI_ir_i)^{-1};$$
$$\Psi_{1i} = EA_i\vartheta_i\lambda^2; \quad \Psi_{2i} = EI_i\rho_iA_i\lambda^2;$$
$$\Psi_{3i} = I_i\rho_is_i^2\lambda^2; \quad \Psi_{4i} = \Psi_{2i} - s_i^4;$$
$$\Psi_{5i} = E(r_iI_i' - N_i'I_i); \quad \Psi_{6i} = \Psi_{2i} + \Psi_{3i} - N_i\Omega_i;$$
$$\Psi_{7i} = (\rho_iN_i^{-1} + r_i\rho_i')A_i\lambda^2; \quad \Psi_{8i} = \eta_is_i^2(r_i\Omega_i + \Psi_{4i});$$
$$\Psi_{9i} = \eta_is_i^2(r_iI_i' - N_i'I_i).$$

Imaginatively extending the structure by one span down and one span up, we write down the conditions of conjugation of adjacent spans of the tower and the boundary conditions at its ends in the form
$$W_i(0) = W_{i-1}(l_{i-1}); \quad \Phi_i(0) = \Phi_{i-1}(l_{i-1});$$
$$\bar{M}_i(0) = \bar{M}_{i-1}(l_{i-1}) - (J_i\lambda^2 + \nu_{\varphi i}\lambda + c_{\varphi i})\Phi_i(0);$$
$$\bar{V}_i(0) = \bar{V}_{i-1}(l_{i-1}) + (m_i\lambda^2 + \nu_{wi}\lambda + c_{wi})W_i(0); \tag{21}$$
$$i = 1,\ 2,\ \dots,\ n;$$
$$\bar{M}_0(l_0) = 0; \bar{V}_0(l_0) = 0;$$
$$\bar{M}_n(0) = 0; \bar{V}_n(0) = 0. \tag{22}$$

Here, $W_0(l_0)$, $\Phi_0(l_0)$, $\bar{M}_0(l_0)$, $\bar{V}_0(l_0)$ and $W_n(0)$, $\Phi_n(0)$, $\bar{M}_n(0)$, $\bar{V}_n(0)$ are the values of deflection, the angle of inclination of the tangent to the bent axis of the structure under the action of bending moments, bending moment and horizontal force at the lower and upper ends of the structure, respectively.

From relations (16), (21), we obtain the following dependence:

$$Y_n(0) = \Psi Y_0(l_0), \tag{23}$$

where $Y_n(0)$ and $Y_0(l_0)$ are column matrices defined according to expression (18); $\Psi$ is the matrix

$$\Psi = B_n \prod_{j=n-1}^{1} F_j^* B_j. \tag{24}$$

The matrices $B_i$ ($i$ = 1, 2, ..., $n$) included in formula (24) have the form

$$B_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b_i & 1 & 0 \\ a_i & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where

$a_i = m_i\lambda^2 + \nu_{\mathrm{w}i}\lambda + c_{\mathrm{w}i},$ . $\quad b_i = J_i\lambda^2 - \nu_{\varphi i}\lambda - c_{\varphi i}$

$F_i^*$ ($i$ = 1, 2, ..., $n$) are matrices determined by formula (17), taking into account relations (19), (20).

## 5. ALGORITHM OF MODAL ANALYSIS OF THE DRILL TOWER

The natural frequencies of the mechanical system are determined in the following sequence. First, we form matrices (17) and (25). When determining the series that are the elements of matrix (17), it is advisable to calculate a certain number of their initial terms and present the constant coefficients in numerical form. Then we find matrix (24), whose elements will also be power series. According to the boundary conditions (22) and expression (23), the frequency equation is obtained by equating to zero the determinant composed of the elements of the third and fourth rows, the first and second columns of matrix (24). The coefficients of the characteristic polynomial are real numbers. Therefore, in general, its roots will be pairs of conjugate complex quantities, the imaginary parts of which will determine the cyclic frequencies of free oscillations of the system.

It should be noted that if the structure is pinched at the base, expression (24) retains its previous form. In this case, the coefficients $a_1$ and $b_1$ included in matrix (25) should be assumed to be zero. The boundary conditions for the anchoring of the lower end are determined by the relations

$$W_0(l_0) = 0; \Phi_0(l_0) = 0.$$

In this case, we use the elements of the third and fourth rows, third and fourth columns of the square matrix (24) to create the frequency equation.

The amplitude functions of the composite rod runs will be the first lines of the matrix relations

$$Y_i(x_i) = F_i^*(x_i) Y_i(0) \quad (i = 1,\ 2,\ \dots,\ n-1\ ). \tag{26}$$

The initial parameters for each of the runs are found using expressions (16), (21). First, we determine the complex quantities $W_0(l_0)$ and $\Phi_0(l_0)$ with an accuracy of a constant factor. To do this, we use the equation obtained by equating the real and imaginary parts of the third and fourth lines of matrix equality (23) to zero. To find each of the forms of oscillations, it is necessary to simultaneously take into account the amplitude functions corresponding to two conjugate complex eigenvalues.

As can be seen from expression (6), the values of the function $u_i(\lambda, x_i)$ for the specified pair of roots of the characteristic equation are also conjugate complex values. Consequently, the corresponding values of the functions $W_i(x_i)$, $\Phi_i(x_i)$, $\bar{M}_i(x_i)$, $\bar{V}_i(x_i)$ will also be conjugate. The free oscillation forms are real functions of the longitudinal coordinate.

The presented method of analyzing dynamic phenomena is oriented to the use of modern computers. In the course of calculations, it becomes necessary to repeatedly solve identical integral equations (7), perform numerous operations with matrices, solve linear algebra problems, find complex roots of characteristic equations, etc. This requires the creation of efficient algorithms for computing processes. The authors of this article have developed software for performing calculations of multi-span drill towers for free and harmonic forced oscillations. The computer program is compiled in a general form and can be used to determine the natural frequencies and forms of free oscillations, as well as the amplitudes of forced oscillations of tower drilling towers with an arbitrary number of spans. The Volterra integral equations of the second kind are solved by the method of successive approximations. Studies have shown that the iterative processes performed in this case have good convergence.

The following section presents the results of calculating the free oscillation frequencies and amplitudes of forced oscillations of a tower-type drilling tower. A significant part of the developed program, designed to generate transition matrices for longitudinal spans with variable parameters along the length, can be used in studies of the dynamics of more complex structures of high-rise structures.

## 6. INFLUENCE OF STRUCTURE PARAMETERS ON FREE OSCILLATION CHARACTERISTICS

When designing drilling towers, there is a need to conduct a comprehensive analysis of the impact of the inertial and stiffness parameters of the composite structure on the frequencies and shapes of its free oscillations. Determining the characteristics of the frequency spectrum makes it possible to avoid resonance phenomena during the operation of drilling rigs. Finding the eigenforms facilitates the study of forced harmonic oscillations of the system by decomposing the waveforms into eigenfunctions. The characteristics of dynamic processes are also useful for calculating the endurance of structural elements.

As an example of the practical application of the proposed method for analyzing transverse oscillations of high-rise structures with variable parameters along their length, consider the results of determining the natural frequencies and oscillation forms of the drilling tower VB 53 × 300.

First, let's consider the simplest case, when the structure is installed on an elastic base and fixed with braces attached to the upper section of the tower. In this case, the design scheme of the structure can be represented as a single-span rod. The initial data for the calculation in the previously adopted notation are given in Table 1. The values of some of these quantities varied within the specified limits. The moment of inertia of the tower cross section was determined by the formula

$$I_1 = (0.738 - 2.217 \cdot 10^{-2}x_1 + 1.664 \cdot 10^{-4}x_1^2)\ m^4.$$

Table 1 Parameters of the drill tower with one tier of braces

| Parameter | Units of measurement | Numerical value | Parameter | Units of measurement | Numerical value |
|---|---|---|---|---|---|
| $E$ | MPa | $2.1\cdot10^5$ | $J_2$ | kg·m² | $2.2\cdot10^3$ |
| $G$ | MPa | $8.1\cdot10^4$ | $c_{w1}$ | N/m | $1.2\cdot10^9$ |
| $L_1$ | M | 53.29 | $c_{w2}$ | N/m | $8.2\cdot10^5$ |
| $x_1$ | - | 0.35 | $c_{\varphi1}$ | N·m/rad | $2.6\cdot10^9$ |
| $F_1$ | m² | $2.95\cdot10^{-2}$ | $c_{\varphi2}$ | N·m/rad | $6.6\cdot10^5$ |
| $\rho_1$ | kg/m³ | $2.04\cdot10^4$ | $\nu_{w1}$ | N·s/m | $5.1\cdot10^5$ |
| $m_1$ | kg | $5.5\cdot10^4$ | $\nu_{w2}$ | N·s/m | 0 |
| $m_2$ | kg | $2.0\cdot10^3$ | $\nu_{\varphi1}$ | N·m·s/rad | $9.4\cdot10^6$ |
| $J_1$ | kg·m² | $8.5\cdot10^4$ | $\nu_{\varphi2}$ | N·m·s/rad | 0 |

Fig. 2 shows the graphs of the first (1), second (2), and third (3) natural frequencies of the drill tower obtained in the absence of longitudinal load on the structure. The dependences show that transverse forces have little effect on the value of the first free oscillation frequency. As the frequency number increases, the influence of shear deformations increases significantly. Thus, when the coefficient $\kappa_1$, which takes into account these deformations, changes from 0.1 to 1, the first free oscillation frequency increases by 5.6%, the second by 39.5%, and the third by 79.3%.

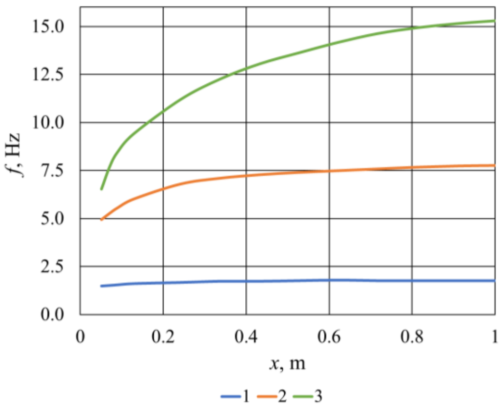


Fig. 2. Dependence of oscillation frequencies on the coefficient characterizing the effect of shear strain

The graphs in Figs. 3 and 4, illustrate the effect of the longitudinal force and stiffness of the braces on the first (curves 1) and second (curves 2) natural frequencies of the tower. The dependences in Fig. 3 are obtained under the assumption of constant axial force along the length of the structure. They show that the static longitudinal loads to which the drill tower is subjected in the operating state have virtually no effect on the natural frequencies of the mechanical system. The dependences in Fig. 4, calculated for $P_1 = 0$ and $c_{\varphi2} = 0$, indicate that the stiffness of the braces largely determines the first frequency of free oscillations. The value of this parameter at higher frequencies is reflected less significantly.

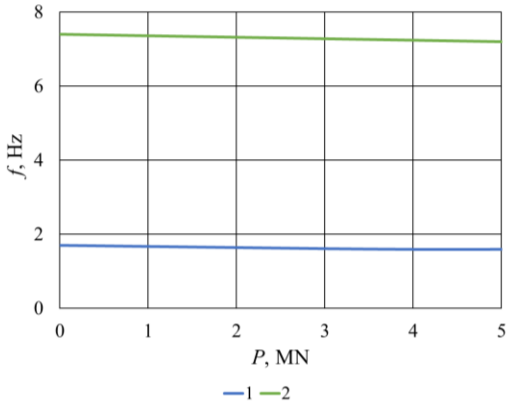


Fig. 3. Dependence of free oscillation frequencies on the function of axial force

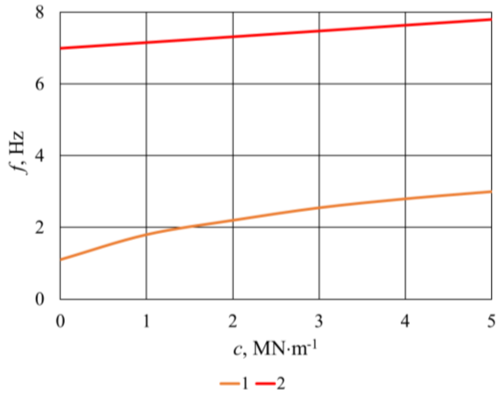


Fig. 4. Dependence of free oscillation frequencies on the stiffness of the braces

To evaluate the effect of the base stiffness on the frequency characteristics of the system, the results of the calculations performed earlier were compared with the results obtained after replacing the elastic base with a clamping base (Table 2).

Table 2 Free oscillation frequencies of the drill tower

| The value of the coefficient $\kappa_1$ | Method of fixing the tower to the base | | | |
|---|---|---|---|---|
| | Elastic support | | Clamping | |
| | Oscillation frequency, Hz | | | |
| | 1 | 2 | 1 | 2 |
| 0.20 | 1.615 | 6.521 | 2.304 | 7.129 |
| 0.35 | 1.637 | 7.093 | 2.421 | 8.271 |
| 0.50 | 1.646 | 7.352 | 2.474 | 8.899 |
| 1.00 | 1.657 | 7.769 | 2.541 | 9.820 |

It was assumed that there was no static longitudinal loading of the tower. As can be seen from the above results, a change in the stiffness of the base has a greater effect on the lower frequency value and a less significant effect on the higher frequency value. As the stiffness of the foundation increases, the effect of shear deformation on the free oscillation frequencies of the structure increases.

Here is an example of a modal analysis of a high-rise structure considered as a two-bay rod of the Timoshenko beam type. We come to this design scheme when determining the natural frequencies and oscillation shapes of a tower with two tiers of bracing. Table 3 shows the mechanical characteristics of the structure, which serve as the initial data for the calculation.

Table 3 Parameters of the drill tower with two tiers of braces

| Parameter | Units of measurement | Numerical value | Parameter | Units of measurement | Numerical value |
|---|---|---|---|---|---|
| $E$ | MPa | $2.1\cdot10^{5}$ | $J_2$ | $kg\cdot m^2$ | 5000 |
| $G$ | MPa | $8.1\cdot10^{4}$ | $J_3$ | $kg\cdot m^2$ | 2000 |
| $l_1$ | m | 35.06 | $c_{w2}$ | N/m | $8.2\cdot10^{5}$ |
| $l_2$ | M | 18.23 | $c_{w3}$ | N/m | $8.2\cdot10^{5}$ |
| $\kappa_1$ | - | 0.35 | $c_{\varphi2}$ | N·m/rad | $6.6\cdot10^{5}$ |
| $\kappa_2$ | - | 0.35 | $c_{\varphi3}$ | N·m/rad | $6.6\cdot10^{5}$ |
| $A_1$ | $m^2$ | $2.952\cdot10^{-2}$ | $\nu_{w2}$ | N·s/m | 0 |
| $A_2$ | $m^2$ | $2.952\cdot10^{-2}$ | $\nu_{w3}$ | N·s/m | 0 |
| $\rho_1$ | $kg/m^3$ | $2.038\cdot10^{4}$ | $\nu_{\varphi2}$ | N·m·s/rad | 0 |
| $\rho_2$ | $kg/m^3$ | $2.038\cdot10^{4}$ | $\nu_{\varphi3}$ | N·m·s/rad | 0 |
| $m_3$ | kg | 2008 | | | |

We solve the problem taking into account the mass of the drill string candles installed in the candlestick. The mass distributed along the length of the candle package is reduced to the junction of the adjacent tower spans. We do not take into account the yielding of the foundation during the calculation. We also assume that the longitudinal force on each of the spans is zero. The expressions of the moments of inertia of the cross-section of the structure for the first and second spans are written in the form

$$I_1 = (0.738 - 2.217\cdot10^{-2}x_1 + 1.664\cdot10^{-4}x_1^2)m^4;$$

$$I_2 = (0.165 - 1.05\cdot10^{-2}x_2 + 1.164\cdot10^{-4}x_2^2)m^4.$$

The results of determining the two lowest natural frequencies of the structure in the form of graphical dependences of these frequencies on the combined mass of the candle package are shown in Fig. 5. As can be seen from the results, the mass of the candles installed in the candlestick significantly affects the natural frequencies of the mechanical system.

The most intense loads that excite transverse oscillations of the tower during well deepening include the forces of interaction between the bent drillpipe string and the rotor table. The frequency of the first harmonic of these loads coincides with the rotation frequency of the string; the frequency of the second harmonic is a multiple of the first frequency and the number of bit cutters. In many cases, high-rise structures of this type are equipped with drilling rigs, the rotor table rotation frequency of which varies in the range of 0.33...3.87 $s^{-1}$ or 0.33...2.7 $s^{-1}$. Comparison of these frequency ranges with the results of determining the tower's natural frequencies presented in Figs. 2-4 and Table 2 shows that in the process of deepening the well, the possibility of resonance phenomena at the main natural frequency of the mechanical system is not excluded.

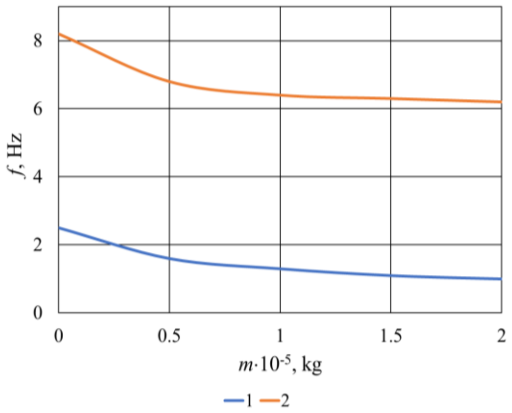


Fig. 5. Dependence of free oscillation frequencies on the combined weight of drillpipes

The conducted studies confirm the need to perform dynamic calculations of drilling towers at the stages of their design and operation. Varying the inertial and rigid characteristics of drill towers makes it possible to change the values of the natural frequencies of the bearing system within a wide range. This contributes to the creation of support structures with optimal parameters, ensuring increased reliability and durability of drilling rigs.

## 7. OSCILLATIONS OF A HIGH-RISE BUILDING UNDER THE INFLUENCE OF DYNAMIC LOADS ON THE BASE

When performing various technological operations related to well deepening, dynamic forces from the winch, rotor, and other mechanisms are transmitted to the tower base elements. Under the influence of these loads, the base and the tower perform joint oscillations. The analysis of these phenomena during the design of drilling rigs helps to select the rational parameters of support structures.

Consider the stationary oscillations of a structure that includes a tower-type structure. Let us assume that the rotor is subjected to horizontal forces from a curved drill string, the upper end of which rotates with some eccentricity. These loads act in the plane of the platform and do not cause significant deformations of the upper part of the base. To a first approximation, the platform and the associated base elements can be treated as a solid body. This example will illustrate the main features of calculating the forced transverse oscillations of a drilling tower.

The design scheme of the mechanical system of the structure is shown in Fig. 1. The transverse oscillations of the structure are described by partial differential equations (1).

The horizontal force acting on the base platform during well deepening is represented by a harmonic function

$$T(t) = T_0 \cos \omega t,$$

where $T_0$ is the amplitude value of the force; ω is the circular frequency; $t$ is time.

We find a partial solution to the problem using the method of complex amplitudes. To do this, we express the dependence of the excitation force on time in a complex form

$$\overline{T}(t) = T_0 e^{\lambda t}$$

Here $e$ *is* the base of the natural logarithm; λ is the value defined by the expression

$$\lambda = i\omega$$

where $i$ is an imaginary unit.

As before, the integrals of equations (1) are found in the form (2). The equations of the amplitude functions (4) are obtained by separating the variables in expressions (1), which leads to the integral equations (5). If the value of λ is known, equation (5) is a Volterra equation of the second kind, which can be solved by the method of successive approximations. However, in order to analyze the amplitude-frequency characteristics of the system, it is advisable to find the solutions of this equation in the form of a dependence on the parameter λ. To do this, we can use the representation of the unknown function $u_i(\lambda, x_i)$ in the form (6). Then the fundamental Cauchy system can be easily determined by the method discussed earlier. The relationship between the geometric and force parameters at the ends of the structure span is expressed by the relation (16).

The conditions for conjugation of neighboring runs, taking into account the reduction of time-dependent terms, are written in the form

$$W_i(0) = W_{i-1}(l_{i-1});\ \Phi_i(0) = \Phi_{i-1}(l_{i-1});$$
$$\overline{M}_i(0) = \overline{M}_{i-1}(l_{i-1}) - \left(J_i\lambda^2 + \nu_{\varphi i}\lambda + c_{\varphi i}\right)\Phi_i(0)$$
$$(i = 1, 2, \ldots, n);$$
$$\overline{Q}_1(0) = \overline{Q}_0(l_0) + (m_1\lambda^2 + \nu_{w1}\lambda + c_{w1})W_1(0) - T_0;$$
$$\overline{Q}_j(0) = \overline{Q}_{j-1}(l_{j-1}) + \left(m_j\lambda^2 + \nu_{wj}\lambda + c_{wj}\right)W_j(0)$$
$$(j = 2, 3, \ldots, n). \tag{27}$$

In this case, the boundary conditions at the ends of the structure are determined by relations (22).

Applying expressions (16) and relation (27), we obtain the matrix dependence

$$Y(0) = \Psi_1 Y_0(l_0) + \Psi_2 T^*, \tag{28}$$

where $Y_0(l_0)$ and $Y_n(0)$ are column matrices of the form (18); $T^*$ is a column matrix

$$T^* = \mathrm{col}(0,0,0; T_0);$$

$\Psi_1$ and $\Psi_2$ are square matrices

$$\Psi_1 = \Psi_2 B_1; \quad \Psi_2 = B_n\left(\prod_{j=n-1}^{2} F_j^* B_j\right)F_1^* . \tag{29}$$

The matrices $F_j^*$ ($j$ = 1, 2, ..., $n$ - 1) and $B_{(j)}$ ($j$ = 1, 2, ..., $n$) included in expressions (29) are defined by relations (17) and (25).

Dependence (28), taking into account the boundary conditions (22), forms a heterogeneous system of algebraic equations with unknowns $W_0(l_0)$, $\Phi_0(l_0)$, $W_n(0)$, $\Phi_n(0)$. Having found the solution to this system, we determine the matrix-columns (18). Using the vector $Y_0(l_0)$, we calculate the initial parameters of each span of the structure using expressions (16) and (27).

The amplitude functions of deflection, cross-sectional angle of rotation, bending moment, and transverse force are determined by matrix equality (26). To find the real values of displacements and internal forces at an arbitrary moment in time, the corresponding amplitude functions should be multiplied by the expression exp($\lambda t$). The real parts of the obtained values are the corresponding physical quantities.

From the above, it can be seen that the algorithms for calculating free and stationary forced transverse oscillations of a structure largely coincide. This feature of the proposed calculation method facilitates its use in engineering practice.

## 8. CONCLUSIONS

1. A mathematical model of free and forced harmonic oscillations of a drill tower as a multi-span Timoshenko beam with bending stiffness, running mass, and axial force variable along the length is developed. Variable parameters may include the elastic moduli of the material and a coefficient that takes into account the beam's shear flexibility. After separating the variables in the equation with partial derivatives and variable coefficients that describe the motion of the high-rise structure, a differential equation of amplitude functions is obtained, which is reduced to an integral equation, the solutions of which are obtained numerically in the form of a power series with respect to the frequency parameter. Sufficient conditions for the convergence of the mentioned series are obtained.
2. Using the matrix initial parameters method, algorithms for calculating free and forced transverse oscillations of a tower are constructed. The influence of shear deformations on the values of the three lower natural frequencies of the mechanical system is investigated. It was found that transverse forces have little effect on the value of the first natural frequency. With the increase of the frequency number, the effect of shear deformations on the frequency value increases significantly. In the considered example, due to the increase in the coefficient $\kappa_1$ from 0.1 to 1, the first, second, and third natural frequencies of the structure increase by 5.6%, 39.5%, and 79.3%, respectively.
   Studies have shown that the static axial load acting on the drill tower during well drilling has virtually no effect on the values of several lower natural frequencies, which indicates that the bending stiffness of the high-rise structure is sufficient. An increase in the stiffness of the braces causes an increase in the values of the free

frequencies of the drill tower. Moreover, the latter factor has a greater impact on the lowest natural frequency of the structure. With an increase in the frequency number, this influence decreases.

3. In the process of performing the technical operation of deepening a well, the drill tower not only holds the drill pipe string that drives the drilling tool in rotation, but also supports the package of drill pipe candles installed in the candlestick. This significantly increases the inertia of the elastic mechanical system, which has a significant impact on its own frequency spectrum.

The most intense loads that excite transverse oscillations of the drill tower include the forces of interaction between the bent drill pipe string and the rotor table. The frequency of the first harmonic of these loads coincides with the rotation frequency of the pipe string, and the second harmonic is a multiple of the specified frequency and the number of bit cutters. Comparison of the excitation frequencies of oscillations with the natural frequency ranges of the drill tower indicates that resonance phenomena may occur during the process of deepening the well, the elimination of which is a necessary condition for ensuring the fatigue strength of the drill pipe string and the fasteners connecting the rotor to the drill tower.

The solution of the problem formulated in the article, as well as the established physical relations, are of significant importance in the field of technical diagnostics of drilling towers.

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