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EFFECTS OF NON-CLASSICAL BOUNDARY CONDITIONS ON THE FREE VIBRATION RESPONSE OF A CANTILEVER EULER-BERNOULLI BEAMS

Abderrachid AFRAS *, **Abdelouafi El GHOULBZOURI**

Laboratory of Applied Sciences, National School of Applied Sciences Al Hoceima, Tetouan, University Abdelmalek Essaadi, Morocco
* Corresponding author, e-mail: abderrachid.afras@gmail.com

Abstract

In this article, the problem of the free vibration behavior of a cantilever Euler-Bernoulli beam with various non-classical boundary conditions, such as rotational, translational spring, and attached mass is investigated. For describing the differential equation of the system. An analytical procedure is proposed firstly, and a numerical method based on the differential transform method DTM is developed in order to validate the obtained results. A parametric study for various degenerate cases is presented with the aim to analyze the influence of rotational stiffness, vertical stiffness, and mass ratio on the free vibration response of the beam, particularly on its modal characteristics. The results show that the non-classical boundary conditions significantly affect the natural frequency and mode shapes of the studied beam system in comparison to the case of a classical boundary conditions such as Simply supported, clamped-clamped, etc. The comparison between the obtained results based on the proposed analytical solution and numerical scheme, and those available in the literature shows an excellent agreement.

Keywords: free vibration, natural frequency, boundary conditions, DTM, non-dimensional stiffness.

Nomenclature
K_T: Vertical spring stiffness;
K_R: Rotational spring stiffness;
M: Masse attached;
L: Length of beam;
E: Young's modulus of materials;
I: Area moment of inertia;
ρ: Beam density;
t: Time;
A: Cross-sectional area;
x: Spatial coordinate;
Z_W: Impedances to shear;
Z_θ: Impedances to bending;
a_x: Dimensionless translational support stiffness;
a_θ: Dimensionless rotational support stiffness;
y (x, t) : Transverse displacement;
R: Mass ration;
β_i : Separation constant;
ω: Circular frequency;
F_w: Shear;
M_θ: Bending moment.

1. INTRODUCTION

Recently, a new class of boundary conditions known as non-classical mechanics systems has drawn considerable attention in the field of engineering, these systems can be used to produce excellent and optimization structural elements in various engineering structures and technologies such as robotic structure, aircraft, vehicles, building, and bridges. The structural response of beams with linear and non-linear elastic boundary conditions has been a topic of many investigations.

Paupitz et al. [1] proposed a new method for calculating the natural frequencies and mode shape by using the terms of dynamic stiffness presented by boundary condition, this method is valid for any linear boundary conditions. Raimondo Luciano et al. [2] studied the free flexural vibrations of nanobeams constrained by non-rigid supports, modelled by transversal and rotational springs. Laila chalah et al. [3] using the finite element method (FEM) for determining the transverse free vibration of a cantilever beam with torsional and translational springs attached at the end, identifying the immediate effects of the elastic restraints on the dynamic behavior of the system. Ding et al. [4] studied the free vibration of an axially moving beam supported by torsional and vertical springs at both ends. the critical speed of the axially moving beam does not change with the vertical spring stiffness. Sayed Mojtaba [5] analysed the free vibration response of a cantilever beam with exponentially varying width and joined by a mass-spring system at the free end. Wang et al. [6] studied the derivation of the frequency equation of flexural vibrating cantilever beam considering the bending moment generated by a mass attached at the free end of the beam The results show that the inertial moment of the mass has the significant effect on the natural frequency and the shape mode. Nuttawit and Arisara Chaikittiratana. [7] applied the differential transform method (DTM) to investigate linear and nonlinear vibration problems of elastically end restrained of functionally graded beams. Suddoung et al. [8] by using the DTM analysed the free vibration stepped response of beams with arbitrary boundary conditions. Dongyan et al. [9] investigated an accurate solution method for free vibration of the Timoshenko beam with general elastic restraints at the end points. Sayed Mojtaba et al. [10] investigated the free vibration analysis of beam with an intermediate sliding connection and joined by a mass-spring system and elastically supported. Banerjee J. R [11]. assembling the dynamic stiffness matrix of the beam and spring-mass system, the Wittrick–Williams algorithm employed to derive natural frequencies and mode shapes of the combined system. The Fourier series has also been used to investigate the free vibration of beams with general restrained boundaries Kim, H. K et al. [12]. Lau, J. H [13]. dealt the vibration Frequencies and mode shapes for a constrained cantilever at some point. Darabi et al. [14] analysed the free vibrations of a beam with a mass-spring system with different boundary conditions both numerically and analytically.

In this study, according to a literature survey and as known that the free vibration's amplitude is the important phenomenon governing the level of vibrations produced in a bridge-beam like structure, the objective is to analyse the free vibration response of a cantilever E-B beam with various non-classical boundary conditions, a problem which can be viewed as a generalization of some cases study presented in the literature [1], [14] in which a developed analytical solution and a numerical method based on the DTM method are used, in order to understand the effect of various boundary conditions on behaviors of the studied beam. The use of the differential transforms method (DTM) in the present analysis is justified by the fact that this has the advantages of rapid convergence and of its effectiveness, to solving nonlinear equations which there is no analytical solution.

2. MECHANICAL MODELS AND PROBLEM FORMULATION

2.1. ANALYTICAL SOLUTION

Consider the configuration represented schematically in Fig. 1, a uniform cantilever Euler-Bernoulli beam attached by mass M and supported at the tip by various boundary conditions, these linear boundary conditions include the translation spring with stiffness K_T, rotational spring with stiffness K_R, the beam is assumed to be of length L and uniform cross-sections. In order to study the variation of the natural frequency and mode shape by taking into account the influence of the boundary conditions, as shown in Fig. 1, seven configuration of boundary condition study are presented. From these examples, some of them have been studied previously and presented here for validation, and we note that the first case is the general cases which combine all the defined non-classical boundary conditions. Determining the characteristic frequency equation described the general system.

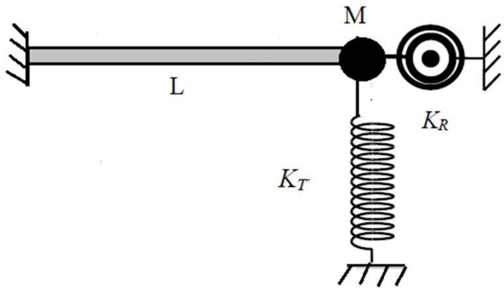

Fig. 1. Scheme of a cantilever Euler-Bernoulli beam attached by mass M and supported by translational springs K_T and rotational springs K_T at the end.

The equation of motion for free vibration of a Euler-Bernoulli beam is:

$$EI\frac{\partial^4 y(X,t)}{\partial x^4} + \rho A\frac{\partial^2 y(X,t)}{\partial t^2} = 0, \quad 0 < X < L \qquad (1)$$

Where y (x, t) is the transverse displacement response at the coordinate measured along the axis of the beam with its origin at its left extremity X and at time t, EI, A, ρ and L are respectively the flexural rigidity, the cross-sectional area, the mass density, and the length of the beam.

For any mode of vibration, the transverse displacement y (x, t) may be written in the form:

$$y(X,t) = W(X)e^{i\omega t} \qquad (2)$$

Where ω is the circular frequency and w(x) is the mode shape, substituting Eq. (2) into Eq. (1) leads to:

$$\frac{\partial^4 W(X)}{\partial X^4} - \frac{\omega^2\rho A}{EI}W(X) = 0 \qquad (3)$$

Eq. (3) can be cast into the dimensionless form as:

$$\frac{\partial^4 w(x)}{\partial x^4} - \beta^4 w(x) = 0 \qquad (4)$$

Where

$$x = \frac{X}{L}, \quad w(x) = \frac{W(X)}{L} \text{ and } \beta^4 = \frac{\omega^2\rho A L^4}{EI}$$

As a function of x, the solution of Eq. (4) can be given by:

$$w(x) == \sigma_1\sin(\beta x) + \sigma_2\cos(\beta x) + \sigma_3\sinh(\beta x) + \sigma_4\cosh(\beta x) \qquad (5)$$

With σ_1, σ_2, σ_3 and σ_4 are shape coefficients which can be found by using the boundary conditions. The beam is clamped at the left end, hence the deflection and slope, Eq. (5) reduce to:

$$w(x) == \sigma_1(\cos(\beta x) - \cosh(\beta x)) + \sigma_2(\sin(\beta x) - \sinh(\beta x) \qquad (6)$$

By using the new approach proposed by Paupitz et al. [1], the boundary conditions are described in terms of dynamic stiffness. The force and moments acting on the beam at the tip are given respectively by:

$$F_w(1) = Z_W w(1),\ M_\theta(1) = Z_\theta w'(1) \quad (7)$$

Where Z_W, and Z_θ represent the dynamic stiffness for lateral and rotator displacement. At the tip, Shear force and bending moment are given by:

$$F_w = EI\frac{\partial^3 w}{\partial x^3},\ M_\theta = EI\frac{\partial^2 w}{\partial x^2} \quad (8)$$

The Boundary condition of the mechanical system in the dimensionless form at the end of the cantilever beam shown in Fig. 1 can be expressed as:

$$\left[\frac{\partial^3 w}{\partial x^3} - \frac{Z_W L^3}{EI} w(x)\right]_{x=1} = 0,$$

$$\left[\frac{\partial^2 w}{\partial x^2} + \frac{Z_\theta L}{EI}\frac{dw}{dx}\right]_{x=1} = 0 \quad (9)$$

Eqs (9) - (6) can be combined to give:

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (10a)$$

Where

$$B_{11} = -(\cos(\beta) + \cosh(\beta)) + \psi_2(\sin(\beta) + \sinh(\beta))$$

$$B_{12} = -(\sin(\beta) + \sinh(\beta)) - \psi_2(\cos(\beta) - \cosh(\beta))$$

$$B_{21} = (\sin(\beta) - \sinh(\beta)) + \psi_1(\cos(\beta) - \cosh(\beta))$$

$$B_{22} = -(\cos(\beta) + \cosh(\beta)) + \psi_1(\sin(\beta) - \sinh(\beta))$$

$$\psi_1 = \frac{Z_W L^3}{EI} = a_X - R\beta^4 \quad ,Z_W = K_T - M\omega^2$$

$$R = \frac{M}{\rho A L} = \frac{\text{attached mass}}{\text{Mass of beam}}$$

$$\psi_2 = \frac{Z_\theta L}{EI} = \frac{k_R L}{EI} = a_\theta \quad (10b)$$

Where a_x and a_θ are dimensionless stiffness parameters translational and rotational respectively and R is the ratio between the additional mass M and the cantilever beam mass which. The frequency equation can be obtained by equating the determinant of Eq.(10a) the coefficients in the above set to zero:

$$1 + \frac{1 + \psi_1\psi_2}{\cos(\beta x)\cosh(\beta x)} - \psi_1(\tan(\beta x - \tanh(\beta x)$$
$$- \tanh(\beta x) + \psi_2(\tanh(\beta x) + \tan(\beta x) - \psi_1\psi_2 = 0 \quad (11)$$

Then, the expression for the mode shape of the n^{th}natural frequencies is given by expression:

$$w_n(x) = \sigma_1((\cos(\beta_n x) - \cosh(\beta_n x) - \left(\frac{\cos(\beta_n) + \cosh(\beta_n) + \psi_2(\sin(\beta_n) + \sinh(\beta_n)}{\sin(\beta_n) + \sinh(\beta_n) - \psi_2(\cos(\beta_n) - \cosh(\beta_n))}\right)(\sin(\beta_n x) - \sinh(\beta_n x))) \quad (12a)$$

the normalized mode shape is defined as:

$$\bar{W}_n(x) = \frac{W_n(x)}{\sqrt{\int_0^1 [W_n(x)]^2 dx}} \quad (12b)$$

2.2. DTM SOLUTION

In order to understand the influence of various boundary conditions as shown in Fig. 1, in the free vibration of the cantilever beam with length L such as variation of natural frequencies and mode shapes, and in addition with the proposed analytical solution presented below, it is of great interest to effectuate an analysis with the Differential Transforms Method (DTM). With its advantage of rapid convergence and easier implantation, the principle of the DTM which is based on the Taylor series expansion, is to transform the governing differential and boundary condition equations into a set of algebraic equations using transformation rules. Tables 1-2 respectively show the basic operation required in differential transformation for the governing differential and boundaries conditions.

Table. 1. Basic operator of DTM for the governing equation, (Nuttawit and Arisara, 2014)

Original function	Transformed function
$f(x) = g(x) \pm \boldsymbol{h}(x)$	$F[k] = G[k] \pm H[k]$
$f(x) = \lambda g(x)$	$F[k] = \lambda G[k]$
$f(x) = g(x)\boldsymbol{h}(x)$	$F[k] = \sum_{l=0}^{r} G[k-l]\, H[l]$
$f(x) = \frac{d^p g(x)}{d^x x}$	$F[k] = \frac{(k+p)}{k} G(k+p)$
$f(x) = x^p$	$F[r] = \delta(k-p) = \begin{cases} 0 r \neq p \\ 1 k \neq p \end{cases}$

Table. 2. Basic operations of DTM for the boundary conditions in dimensionless forms

x=0		x=1	
Original B.C	Transformed B.C	Original B.C	Transformed B.C
$f(0) = 0$	$F(0) = 0$	$f(1) = 0$	$\sum_{k=0}^{\infty} F(k) = 0$
$\frac{df(0)}{dx} = 0$	$F(1) = 0$	$\frac{df(1)}{dx} = 0$	$\sum_{k=0}^{\infty} k\,F(k) = 0$
$\frac{d^2 f(0)}{d^2 x} = 0$	$F(2) = 0$	$\frac{d^2 f(1)}{d^2 x} = 0$	$\sum_{k=0}^{\infty} k(k-1)\,F(k) = 0$
$\frac{d^3 f(0)}{d^3 x} = 0$	$F(3) = 0$	$\frac{d^3 f(1)}{d^3 x} = 0$	$\sum_{r=0}^{\infty} k(k-1)(k-2)\,F(k) = 0$

The general function f(x) in Tables 1-2 is considered as the transversal displacement w(x), we apply the basic operation of DTM presented in Eq. (4) we obtained the recurrence equation as:

$$w[k+4] = \frac{\beta^4}{(k+1)(k+2)(k+3)(k+4)} w[k] \quad (13)$$

Where

$$\beta^4 = \frac{\omega^2 \rho A L^4}{EI}$$

Let the non-zero values of shear force and bending moment indicates by Ω_1 and Ω_2, with applying the basic operations of DTM for the boundary condition at x=0, and by using the Table. 2, once we obtain:

$$w[0] = 0, w[1] = 0, w[2] = \Omega_1, w[3] = \Omega_2 \quad (14)$$

Substituting Eq. (14) into the recurrence Eq. (13) leads to $w\ [k]$ for all values of k as follows:

$$w[4k] = 0, \quad k = 0,1,2,3,\ldots \quad (15a)$$

$$w[4k+1] = 0, \ k = 0,1,2,3,\ldots \quad (15b)$$

$$w[4k+2] = \frac{\beta^{4k}}{(4k+2)!}\Omega_1, \quad k = 0,1,2,3,\ldots \quad (15c)$$

$$w[4k+3] = \frac{\beta^{4k}}{(4k+3)!}\Omega_2, \ k = 0,1,2,3,\ldots \quad (15d)$$

For the boundary conditions x=1 as presented in Eq. (9), by applying the basic operations of DTM presented in Table. 2, and by using the terms of dynamic stiffness Eq. (7), one obtains:

$$\sum_{r=0}^{\infty} k(k-1)\, w[k] + \psi_2 \sum_{k=0}^{\infty} k\, w[k] = 0$$

$$\sum_{k=0}^{\infty} k(k-1)(k-2)\, w[k] - \psi_1 \sum_{k=0}^{\infty} w[k] = 0 \quad (16)$$

Substituting the expression w[k] from Eq. (16) into Eq. (14) leads to two polynomial equations which can be arranged into the matrix form:

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (17a)$$

Where the elements in the matrix are:

$$D_{11} = \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k-1)!} - \psi_1 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+2)!}$$

$$D_{12} = \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k)!} - \psi_1 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+3)!}$$

$$D_{21} = \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+1)!}$$

$$D_{22} = \sum_{k=1}^{R} \frac{\beta^{4k}}{(4k+1)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+2)!} \quad (17b)$$

The frequency equation is:

$$\left(\sum_{k=0}^{R} \frac{\beta^{4k}}{(4k-1)!} - \psi_1 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+2)!}\right)$$

$$\left(\sum_{k=1}^{R} \frac{\beta^{4k}}{(4k+1)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+2)!}\right)$$

$$-\left(\sum_{k=0}^{R} \frac{\beta^{4k}}{(4k)!} - \psi_1 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+3)!}\right)$$

$$\left(\sum_{k=0}^{R} \frac{\beta^{4k}}{(4k)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta^{4k}}{(4k+1)!}\right) = 0 \quad (18)$$

Solving the frequency equation. Eqs. (11) and (18) by using the algorithm of Newton Raphson programmed in MATLAB environment, and by solving the Eq. (16), one can obtain the frequency values in the flowing form $\beta = \beta_k^{[\epsilon]}$, where k=1,2,3,…,ϵ, in which $\beta_k^{[\epsilon]}$ is the k[th] estimated frequency corresponding to ϵ, hence, an appropriate value of ϵ is obtained by convergence analysis with the following equation $\beta_k^{[\epsilon]} - \beta_k^{[\epsilon-1]} \leq \lambda$ where λ is a given error tolerance. The mode shape function can be obtained by using the expression:

$$w(x) = \sum_{k=0}^{R} x^k w[k]$$

$$w(x) = \sum_{k=0}^{R} \frac{\beta_n{}^{4k}}{(4k+2)!} x^{(4k+2)} + \Omega_2 \sum_{k=0}^{R} \frac{\beta_n{}^{4k}}{(4k+3)!} x^{(4k+3)}$$

Where

$$\Omega_2 = -\frac{\sum_{k=0}^{R} \frac{\beta_n{}^{4k}}{(4k)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta_n{}^{4k}}{(4k+1)!}}{\sum_{k=1}^{R} \frac{\beta_n{}^{4k}}{(4k+1)!} + \psi_2 \sum_{k=0}^{R} \frac{\beta_n{}^{4k}}{(4k+2)!}} \quad (19)$$

Where ψ_2 is defined in the Eq. (10b).

3. NUMERICAL RESULTS AND DISCUSSION

In this section, with the aim of analysing the effect of various boundary conditions in the free vibration response of a cantilever beam as shown in Fig.1, and to show the versatility of the aforementioned theory, like the analytical and numerical solution, six examples are given to illustrate this, and are presented as a simplification referring to the example 1. In addition, most parameters affect the frequency and mode shape of the specified case, are considered in term of non-dimensional ratios, such as non-dimensional translational stiffness a_x, rotational stiffness a_θ, and non-dimensional mass R, these ratios are defined in Eq. (10b).

Example 1: cantilever beam with rotational, translation spring and mass restraint at the tip

By solving Eq. (11) and (18), we can obtain the values of three first dimensionless natural frequencies, which are listed in Table3 **(Appendix 1**). Fig. 2, shows the variation of the first three dimensionless frequencies as a function of the kth eigenvalues defined by the Eq. (18) corresponding to the values $R=a_x=a_\theta=100$**.** From this figure, we can observe that dimensionless natural frequencies determined by the differential transform method converge very rapidly, in which for the first non-dimensional frequency the convergence is attained for k=1, and for the second and third non-dimensional frequency convergence is attained for k=2, and for k=4 respectively. The convergence is assured in terms of the third non-dimensional frequency, consequently this is one of the reasons why the DTM is used in this work. Examining Fig. 3, which represents the first three mode shapes of the beam with fixed values $R=a_x=a_\theta=100$ and a comparison between the analytical solution and DTM solution, it can be seen that there is an excellent agreement between the two solutions, which shows the accuracy of the proposed solution. Fig. 4 shows the first three dimensionless natural frequencies $\beta_{i=1,2,3}$ of a beam with various parameters of R, a_x and a_θ. It is observed that the effect of the value of R, a_x and a_θ on the lower mode is more significant, and can be drawn, particularly for the fundamental mode, that when dimensionless stiffness $a_x=a_\theta$ keep constant and the mass ratios increase, the dimensionless natural frequencies β_i of system decreases, when dimensionless stiffness $a_x=a_\theta$ increase and the mass ratios R .
keep constant, the dimensionless natural frequencies β_i increase, at the lower modes are more sensitive to the boundary restraints then the natural frequencies at the higher modes, stiffness parameter, and mass ratios give significant change on the natural frequencies.

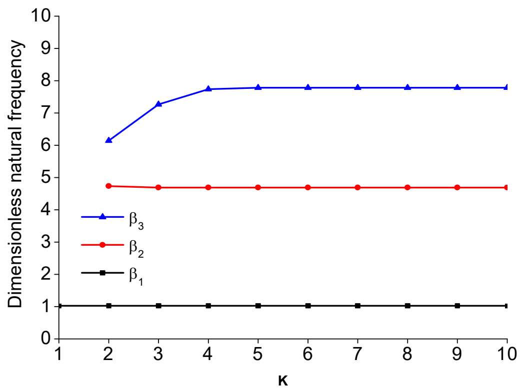

Fig. 2. Convergence of the first, second and third natural frequencies with $R=a_x=a_\theta=100$

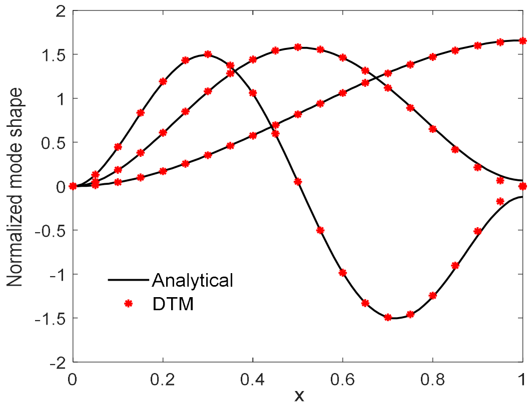

Fig. 3. The 1st to 3th mode shape of a beam with $R=a_x=a_\theta=100$, comparison between the analytical solution and DTM solution

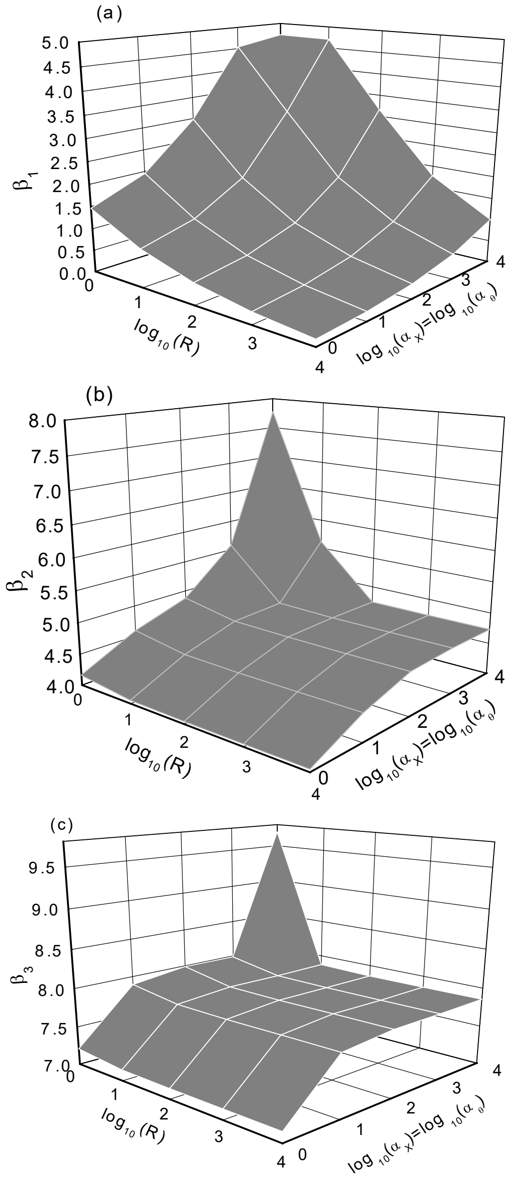

Fig. 4. The first three dimensionless natural frequency of a beam with various parameters R, a_x and a_θ

Example 2: Cantilever beam with point mass and translational spring

In the present case, and referring to the general case, the rotational spring is neglected a_θ, which a variation of the natural frequencies can be produced. Table 4 (Appendix 1) shows the variation of the first three dimensionless natural frequencies as a function of the non-dimensional mass coefficient R and the non-dimensional stiffness coefficient a_x, which are obtained from Eqs. (11) and (18) and includes a comparison between the results proved by Kim, H. K et al. [12], which they, in their paper, treated the same problem by using the Fourier series. From this table, one can notice clearly that the obtained results are in good agreement with those obtained by using the Fourier transform, with a slight error, the think which explain the accuracy of the presented methods. In addition, other conclusions also can be drowned for a fixed value of R, as the non-dimensional stiffness coefficient a_x increases the dimensionless natural frequencies βi increase, Inversely, for a fixed value of ax, as the non-dimensional mass coefficient R increases, the dimensionless natural frequencies βi decreases.

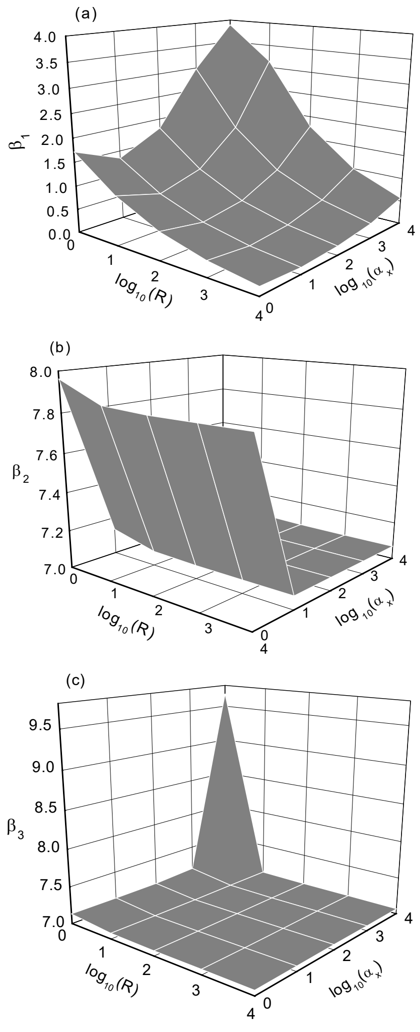

Fig. 5. The first three dimensionless natural frequency of a beam with various parameters R, a_x.

Example 3: Cantilever beam joined by mass and rotational spring at the tip

In this case, comparably with the case 2, only the linear rotational spring and the mass are considered, and the vertical spring is neglected a_x =0, as there are no results in term of frequency in the literature. Table 5 **(Appendix 1**). presents the values of dimensionless natural frequencies $\beta_{i=1,2,3}$ for a various value of a_θ, and R. From this table, especially for the first natural frequency, can be drown that for a fixed value of R, as the non-dimensional stiffness coefficient a_θ increases the dimensionless natural frequencies β_i increase, Inversely, for a fixed value of a_θ, as the non-dimensional mass coefficient R increases, the dimensionless natural frequencies β_i decreases.

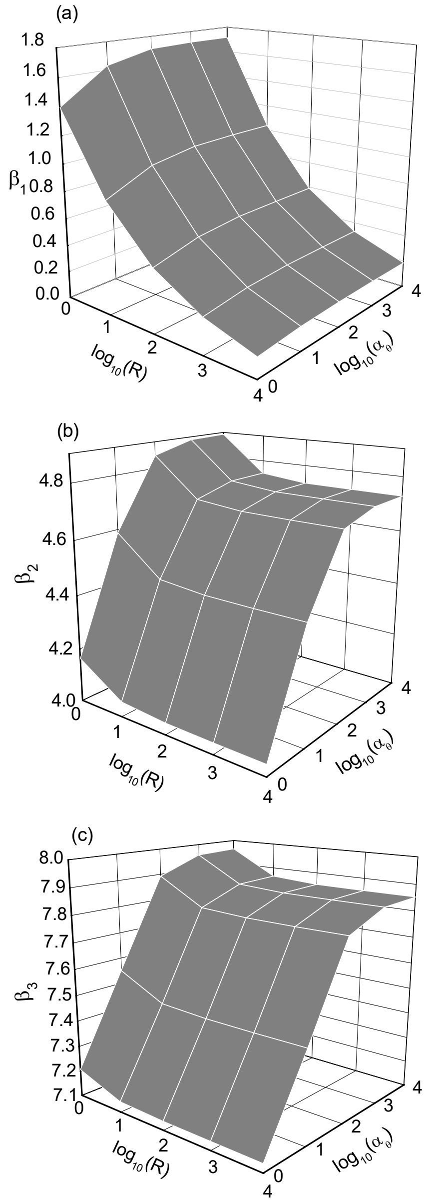

Fig. 6. The first three dimensionless natural frequency of a beam with various parameters R, a_θ

Example 4: cantilever beam joined by spring system at the tip

In this case, beam with translational, rotational spring, and no mass attached spring boundary condition R=0. The dimensionless natural frequencies, in this case, are provided in Table. 6 **(Appendix 1**) By comparing the present results with the available results given by Lau et al [13], the good agreement demonstrate clearly the accuracy of the present solutions. From Fig. 7, which represents the variation of the first three dimensionless frequencies as a function of a_x, a_θ the most conclusions that can be drawn are when dimensionless stiffness a_θ keep constant and the a_x increase, the natural frequency of system increases, also when dimensionless stiffness a_θ increase and the a_x keep constant, the natural frequency of system increases. If a_x and a_θ approach infinity, the beam becomes a clamped-clamped.

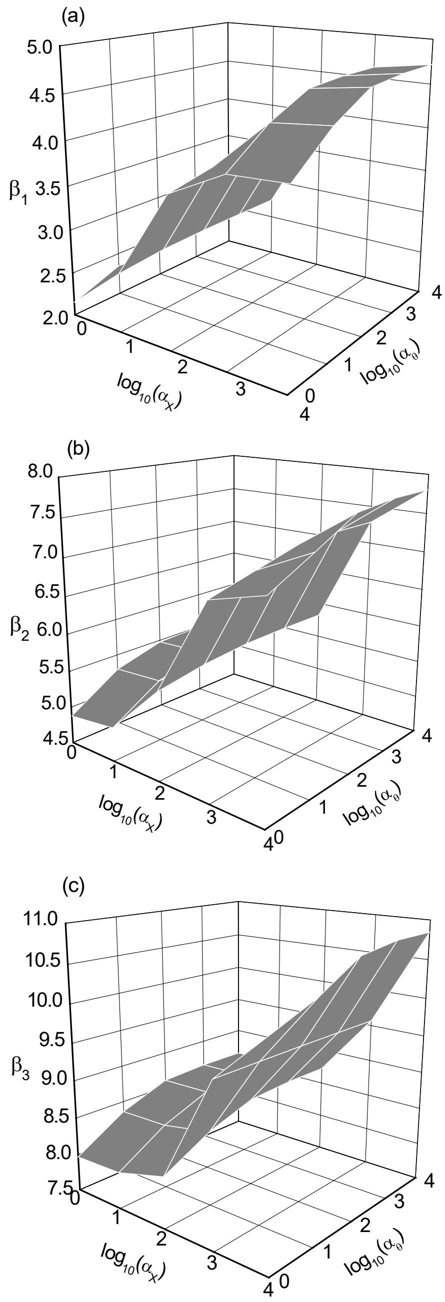

Fig. 7. The first three dimensionless natural frequency of beam with various parameters a_x ,a_θ

Example 5: Cantilever beam joined by translational spring at the tip

We consider the beam with translational spring. The dimensionless natural frequencies in this case are provided in Table. 7 **(Appendix 1**). Comparing the present's results with those available given by Banerjee. [11], in which in his work the dynamic stiffness matrix and the Wittrick – Williams algorithm to derive the natural frequencies of a similar system, demonstrate the accuracy of the proposed analytical solution. From Fig. 8, which illustrates the variation of the first three dimensionless natural frequencies as a function of a_x, important conclusions can be drawn that when dimensionless stiffness a_x increase, the natural frequency of system increases too, Fig. 9 shows the variation of the first three mode shape with various value of a_x. It can be seen from the results that the non dimensionless parameter a_x has a significant effect on the natural frequencies and mode shape, if a_x approaches to infinity the beam becomes clamped-pinned system.

Example 6: Cantilever beam joined by rotation spring at the tip

In this case, the cantilever beam is connected only by a rotational spring at the end, the effect of the vertical spring and mass are neglected, to show the influence of the rotational spring with non-dimensional stiffness a_θ, Table 8 **(Appendix 1)** represents the variation of the first three dimensionless natural frequency as a function of a_θ and a comparison between the present results and the results presented by Lau et al. [13]. From this table and fig.12, the accuracy of the proposed solution proposed has been proved, and one can conclude that there is a slow effect in dimension-

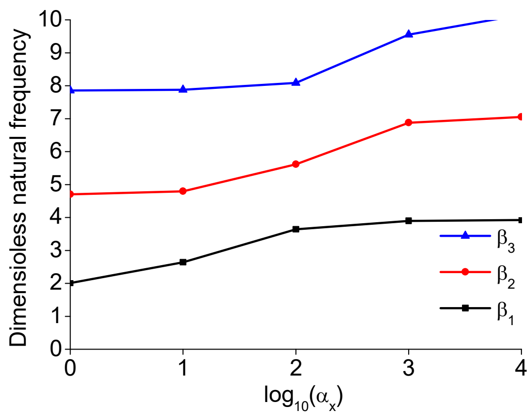

Fig. 8. The first three dimensionless natural frequency of beam with various parameters a_x

less natural frequencies caused by the presence of this rotational spring at the beam's end for slow values of a_θ, the think which creates a difference between this case and the case 5 where only the vertical spring is introduced.

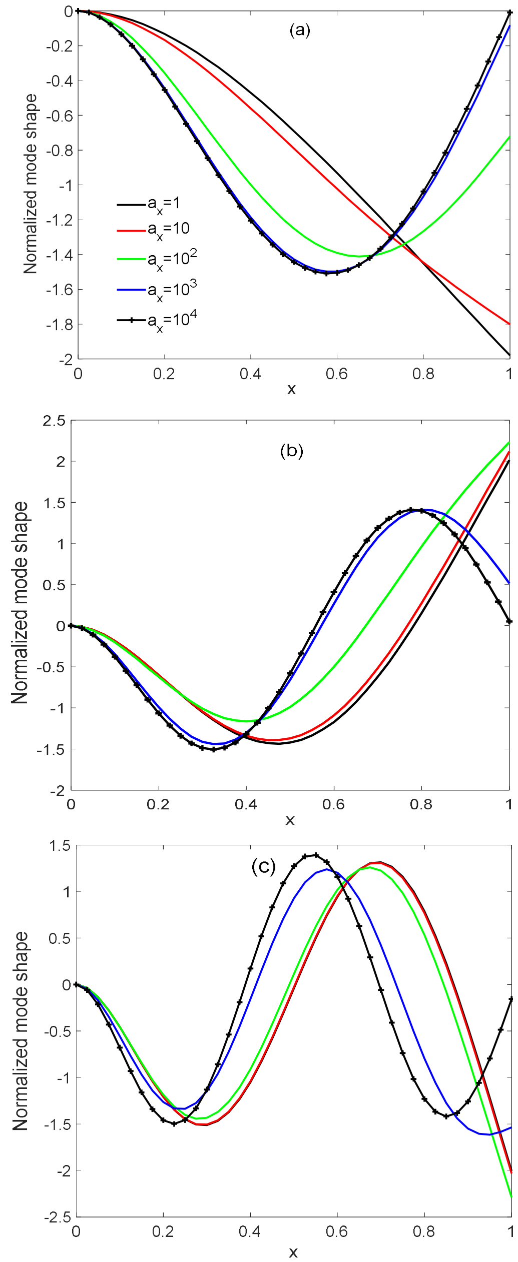

Fig. 9. The first three mode shape of beam with various values of a_x

From fig. 12, represents the variation of the first dimensionless natural frequency as a function of the non-dimensional stiffness ratio a_θ, a_x, one can notice that for a specified value of these parameters, with the present calculation and examples, can be obtained for any value of a_θ, a_x and i.e. for pinned, clamped and free condition and one can say that the powerful of the presented solutions reside in its higher ability to give an idea about different system of boundary conditions in reality.

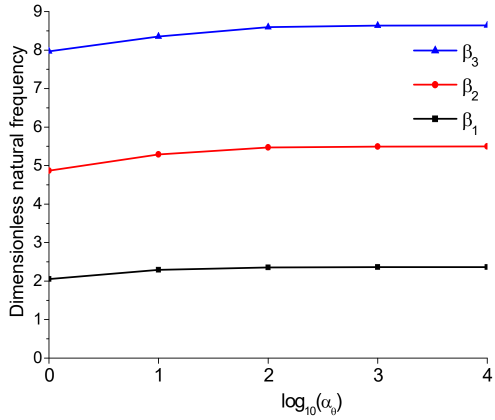

Fig. 10. The first three dimensionless natural frequency of beam with various parameters a_θ

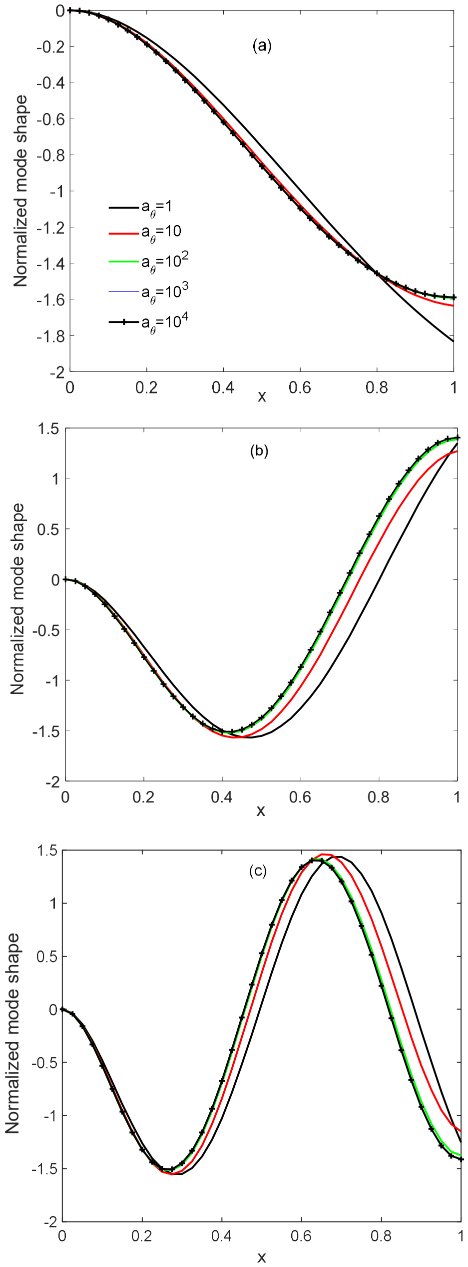

Fig. 11. The first three mode shape of beam with various values of a_θ

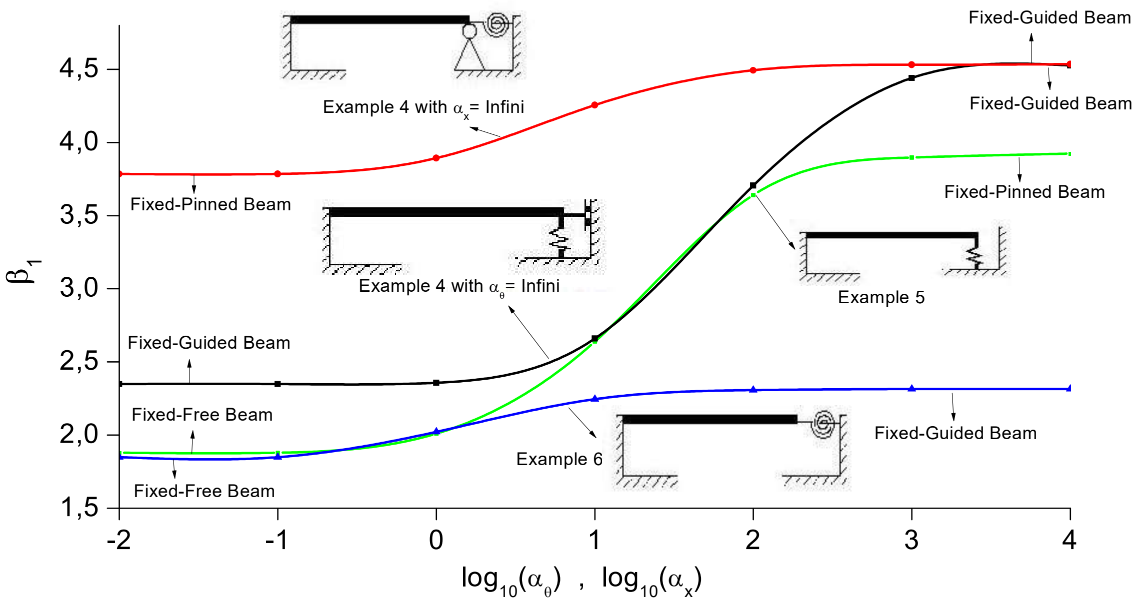

Fig.12. Variation of the first dimensionless natural frequency parameter as a function of a_θ, a_x

CONCLUSIONS

The free vibration analysis of a cantilever Euler-Bernoulli beams with various non-classical linear boundary conditions at the tip, such as a linear rotational and vertical springs and a concentrated mass has been investigated in this study. The effect of these systems on the modal characteristics such as natural frequencies and mode shapes of the beam in different configurations is demonstrated by two methods, where a developed analytical solution and a numerical method based on the DTM are used. With a parametric study, the results show an excellent agreement with the previous published results and the two presented solutions proved their performance, and the expressions for the natural frequencies and mode shapes for the aforementioned boundary conditions are determined. Consequently, the results indicate that the present paper can be used very effectively in the design of a beam with various non-classical supporting conditions.

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Appendix 1

Table. 3. The first three dimensionless natural frequencies of a beam with various parameters R, a_x and a_θ

$a_x = a_\theta$	frequency coefficients	Method	R				
			1	10	10^2	10^3	10^4
1	β_1	Analytical	1.4641	0.8671	0.4904	0.2759	0.1551
		DTM	1.4641	0.8671	0.4904	0.2759	0.1551
	β_2	Analytical	4.1629	4.0556	4.0432	4.0419	4.0418
		DTM	4.1629	4.0556	4.0432	4.0419	4.0418
	β_3	Analytical	7.2064	7.1416	7.1346	7.1339	7.1338
		DTM	7.2064	7.1416	7.1346	7.1339	7.1338
10	β_1	Analytical	1.9530	1.1710	0.6633	0.3733	0.2099
		DTM	1.9530	1.1710	0.6633	0.3733	0.2099
	β_2	Analytical	4.5908	4.4489	4.4321	4.4304	4.4302
		DTM	4.5908	4.4489	4.4321	4.4304	4.4302
	β_3	Analytical	7.5502	7.4608	7.4509	7.4499	7.4498
		DTM	7.5502	7.4608	7.4509	7.4499	7.4498
10^2	β_1	Analytical	2.9875	1.8114	1.0270	0.5780	0.3250
		DTM	2.9875	1.8114	1.0270	0.5780	0.3250
	β_2	Analytical	4.8755	4.7055	4.6873	4.6854	4.6852
		DTM	4.8755	4.7055	4.6873	4.6854	4.6852
	β_3	Analytical	7.8952	7.7936	7.7824	7.7812	7.7811
		DTM	7.8952	7.7936	7.7824	7.7812	7.7811

10^3	β_1	Analytical	4.5407	3.1396	1.7819	1.0028	0.5640
		DTM	4.5407	3.1396	1.7819	1.0028	0.5640
	β_2	Analytical	5.5830	4.7499	4.7274	4.7255	4.7253
		DTM	5.5830	4.7499	4.7274	4.7255	4.7253
	β_3	Analytical	7.9870	7.8583	7.8466	7.8455	7.8454
		DTM	7.9870	7.8583	7.8466	7.8455	7.8454
10^4	β_1	Analytical	4.7187	4.7093	3.1599	1.7786	1.0002
		DTM	4.7187	4.7093	3.1599	1.7786	1.0002
	β_2	Analytical	7.7726	5.6179	4.7321	4.7297	4.7295
		DTM	7.7726	5.6179	4.7321	4.7297	4.7295
	β_3	Analytical	9.6953	7.8693	7.8537	7.8525	7.8524
		DTM	9.6953	7.8693	7.8537	7.8525	7.8524

Table 4 The first three dimensionless natural frequency of beam with various parameters a_x and R

a_x	Frequency coefficients	Method	R				
			1	10	10^2	10^3	10^4
1	β_1	Analytical	1.3408	0.7906	0.4469	0.2514	0.1414
		DTM	1.3408	0.7906	0.4469	0.2514	0.1414
		Kim et al [12]	1.3413	0.7908	0.4471	0.2514	0.1415
	β_2	Analytical	4.0314	3.9384	3.9278	3.9267	3.9266
		DTM	4.0314	3.9384	3.9278	3.9267	3.9266
		Kim et al [12]	4.0331	3.9400	3.9293	3.9281	3.9281
	β_3	Analytical	7.1341	7.0756	7.0692	7.0686	7.0685
		DTM	7.1341	7.0756	7.0692	7.0686	7.0685
		Kim et al [12]	7.1367	7.0781	7.0718	7.0712	7.0711
10	β_1	Analytical	1.7988	1.0615	0.6001	0.3376	0.1898
		DTM	1.7988	1.0615	0.6001	0.3376	0.1898
		Kim et al [12]	1.7992	1.0617	0.6002	0.3377	0.1899
	β_2	Analytical	4.0346	3.9385	3.9278	3.9267	3.9266
		DTM	4.0346	3.9385	3.9278	3.9267	3.9266
		Kim et al [12]	4.0362	3.9400	3.9293	3.9283	3.9281
	β_3	Analytical	7.1343	7.0756	7.06929	7.0686	7.0685
		DTM	7.1343	7.0756	7.06929	7.0686	7.0685
		Kim et al [12]	7.1369	7.0781	7.0718	7.0712	7.0711
10^2	β_1	Analytical	2.9842	1.7808	1.0068	0.5664	0.3185
		DTM	2.9842	1.7808	1.0068	0.5664	0.3185
		Kim et al [12]	2.9847	1.7809	1.0068	0.5665	0.3186
	β_2	Analytical	4.0788	3.9389	3.9278	3.9267	3.9266
		DTM	4.0788	3.9389	3.9278	3.9267	3.9266
		Kim et al [12]	4.0803	3.9405	3.9293	3.9283	3.9281
	β_3	Analytical	7.1365	7.0756	7.0692	7.0686	7.0685
		DTM	7.1365	7.0756	7.0692	7.0686	7.0685
		Kim et al [12]	7.1391	7.0782	7.0718	7.0712	7.0711
10^3	β_1	Analytical	3.8892	3.1396	1.7785	1.0006	0.5627
		DTM	3.8892	3.1396	1.7785	1.0006	0.5627
		Kim et al [12]	3.8907	3.1397	1.7785	1.0007	0.5628
	β_2	Analytical	5.5007	3.9465	3.9278	3.9267	3.9266
		DTM	5.5007	3.9465	3.9278	3.9267	3.9266
		Kim et al [12]	5.5014	3.9481	3.9294	3.9283	3.9281
	β_3	Analytical	7.1693	7.0759	7.0692	7.0686	7.0685
		DTM	7.1693	7.0759	7.0692	7.0686	7.0685
		Kim et al [12]	7.1718	7.0782	7.0718	7.0712	7.0711
10^4	β_1	Analytical	3.9236	3.9228	3.1599	1.7783	1,0000
		DTM	3.9236	3.9228	3.1599	1.7783	1,0000
		Kim et al [12]	3.9252	3.9244	3.1600	1.7783	1,0001
	β_2	Analytical	7.0447	5.6094	3.9286	3.9267	3.92661
		DTM	7.0447	5.6094	3.9286	3.9267	3.92661
		Kim et al [12]	7.0473	5.6095	3.9302	3.9283	3.9281
	β_3	Analytical	9.6787	7.0802	7.0693	7.0686	7.0685
		DTM	9.6787	7.0802	7.0693	7.0686	7.0685
		Kim et al [12]	9.6803	7.0827	7.0718	7.0712	7.0711

Table. 5. The first three dimensionless natural frequency of beam with various parameters a_θ and R

a_θ	Frequency coefficients	method	R				
			1	10	10^2	10^3	10^4
1	β_1	Analytical	1.3966	0.8270	0.4677	0.2631	0.14801
		DTM	1.3966	0.8270	0.4677	0.2631	0.14801
	β_2	Analytical	4.1626	4.0556	4.04323	4.0419	4.04184
		DTM	4.1626	4.0556	4.04323	4.0419	4.04184
	β_3	Analytical	7.2064	7.1416	7.13462	7.1339	7.1338
		DTM	7.2064	7.1416	7.13462	7.1339	7.1338
10	β_1	Analytical	1.6312	0.9774	0.5536	0.3115	0.1752
		DTM	1.6312	0.9774	0.5536	0.3115	0.1752
	β_2	Analytical	4.5878	4.4489	4.43216	4.4304	4.4302
		DTM	4.5878	4.4489	4.43216	4.4304	4.4302
	β_3	Analytical	7.5499	7.4608	7.45096	7.4499	7.4498
		DTM	7.5499	7.4608	7.45096	7.4499	7.4498
10^2	β_1	Analytical	1.7081	1.0297	0.5837	0.3285	0.1847
		DTM	1.7081	1.0297	0.5837	0.3285	0.1847
	β_2	Analytical	4.8486	4.7051	4.6873	4.6854	4.6852
		DTM	4.8486	4.7051	4.6873	4.6854	4.6852
	β_3	Analytical	7.8926	7.7936	7.7824	7.7812	7.7811
		DTM	7.8926	7.7936	7.7824	7.7812	7.7811
10^3	β_1	Analytical	1.7177	1.0363	0.5875	0.3306	0.1859
		DTM	1.7177	1.0363	0.5875	0.3306	0.1859
	β_2	Analytical	4.8881	4.7452	4.7273	4.7255	4.7253
		DTM	4.8881	4.7452	4.7273	4.7255	4.7253
	β_3	Analytical	7.9567	7.8579	7.8466	7.8455	7.8454
		DTM	7.9567	7.8579	7.8466	7.8455	7.8454
10^4	β_1	Analytical	1.7187	1.037	0.5879	0.3309	0.1861
		DTM	1.7187	1.037	0.5879	0.3309	0.1861
	β_2	Analytical	4.8923	4.7494	4.7316	4.7297	4.7295
		DTM	4.8923	4.7494	4.7316	4.7297	4.7295
	β_3	Analytical	7.9636	7.8649	7.8536	7.8525	7.8524
		DTM	7.9636	7.8649	7.8536	7.8525	7.8524

Table. 6. The first three dimensionless natural frequencies of a beam with various parameters a_x a_θ,

a_θ			a_x				
			1	10	10^2	10^3	10^4
1	β_1	Analytical	2.1490	2.6662	3.6818	4.00421	4.0380
		DTM	2.1490	2.6662	3.6818	4.00421	4.0380
		Lau et al [13]	2.1490	2.6662	3.6818	4.00421	4.0380
	β_2	Analytical	4.8767	4.9520	5.6517	6.9139	7.1132
		DTM	4.8767	4.9520	5.6517	6.9139	7.1132
		Lau et al [13]	4.8767	4.9520	5.6517	6.9139	7.1132
	β_3	Analytical	7.9676	7.9851	8.1759	9.5537	10.1957
		DTM	7.9676	7.9851	8.1759	9.5537	10.1957
		Lau et al [13]	7.9676	7.9851	8.1759	9.5537	10.1957
10	β_1	Analytical	2.3470	2.7146	3.7888	4.3562	4.4229
		DTM	2.3470	2.7146	3.7888	4.3562	4.4229
		Lau et al [13]	2.3470	2.7146	3.7888	4.3562	4.4229
	β_2	Analytical	5.2932	5.3348	5.7561	7.0843	7.4153
		DTM	5.2932	5.3348	5.7561	7.0843	7.4153
		Lau et al [13]	5.2932	5.3348	5.7561	7.0843	7.4153
	β_3	Analytical	8.3543	8.3660	8.4888	9.5593	10.4274
		DTM	8.3543	8.3660	8.4888	9.5593	10.4274
		Lau et al [13]	8.3543	8.3660	8.4888	9.5593	10.4274
10^2	β_1	Analytical	2.4036	2.7309	3.8403	4.5845	4.6754
		DTM	2.4036	2.7309	3.8403	4.5845	4.6754
		Lau et al [13]	2.4036	2.7309	3.8403	4.5845	4.6754
	β_2	Analytical	5.4740	5.5029	5.8130	7.2521	7.7334
		DTM	5.4740	5.5029	5.8130	7.2521	7.7334
		Lau et al [13]	5.4740	5.5029	5.8130	7.2521	7.7334
		Analytical	8.5990	8.6067	8.6870	9.5649	10.7605

	β_3	DTM	8.5990	8.6067	8.6870	9.5649	10.7605
		Lau et al [13]	8.5990	8.6067	8.6870	9.5649	10.7605
10^3	β_1	Analytical	2.4104	2.7329	3.8474	4.6204	4.7151
		DTM	2.4104	2.7329	3.8474	4.6204	4.7151
		Lau et al [13]	2.4104	2.7329	3.8474	4.6204	4.7151
	β_2	Analytical	5.4980	5.5253	5.8212	7.2846	7.7957
		DTM	5.4980	5.5253	5.8212	7.2846	7.7957
		Lau et al [13]	5.4980	5.5253	5.8212	7.2846	7.7957
	β_3	Analytical	8.6358	8.6429	8.7172	9.5661	10.8401
		DTM	8.6358	8.6429	8.7172	9.5661	10.8401
		Lau et al [13]	8.6358	8.6429	8.7172	9.5661	10.8401
10^4	β_1	Analytical	2.4111	2.7331	3.8482	4.6242	4.7193
		DTM	2.4111	2.7331	3.8482	4.6242	4.7193
		Lau et al [13]	2.4111	2.7331	3.8482	4.6242	4.7193
	β_2	Analytical	5.5005	5.5276	5.8221	7.2881	7.8025
		DTM	5.5005	5.5276	5.8221	7.2881	7.8025
		Lau et al [13]	5.5005	5.5276	5.8221	7.2881	7.8025
	β_3	Analytical	8.6397	8.6467	8.7204	9.5662	10.8491
		DTM	8.6397	8.6467	8.7204	9.5662	10.8491
		Lau et al [13]	8.6397	8.6467	8.7204	9.5662	10.8491

Table. 7. The first three dimensionless natural frequencies of a beam with various parameters a_x

Frequency coefficients	Method	a_x				
		1	10	10^2	10^3	10^4
β_1	Analytical	2.0100	2.6389	3.6405	3.8978	3.9237
	DTM	2.0100	2.6389	3.6405	3.8978	3.9237
	Banerjee J. R [11]	2.0100	2.6389	3.6406	3.8978	-
β_2	Analytical	4.7037	4.7937	5.6159	6.8762	7.0507
	DTM	4.7037	4.7937	5.6159	6.8762	7.0507
	Banerjee J. R [11]	4.7037	4.7937	5.6160	6.8762	-
β_3	Analytical	7.8568	7.8756	8.0840	9.5525	10.1549
	DTM	7.8568	7.8756	8.0840	9.5525	10.1549
	Banerjee J. R [11]	7.8568	7.8756	8.0840	9.5525	-

Table. 8. The first three dimensionless natural frequencies of a beam with various parameters a_θ

Frequency coefficients	Method	a_θ				
		1	10	10^2	10^3	10^4
β_1	Analytical	2.0539	2.2911	2.3564	2.3641	2.3649
	DTM	2.0539	2.2911	2.3564	2.3641	2.3649
	Lau et al [13]	2.0539	2.2911	2.3564	2.3641	2.3649
β_2	Analytical	4.8686	5.2887	5.4708	5.4950	5.4975
	DTM	4.8686	5.2887	5.4708	5.4950	5.4975
	Lau et al [13]	4.8686	5.2887	5.4708	5.4950	5.4975
β_3	Analytical	7.9656	8.3530	8.5982	8.6350	8.6389
	DTM	7.9656	8.3530	8.5982	8.6350	8.6389
	Lau et al [13]	7.9656	8.3530	8.5982	8.6350	8.6389

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Abderrachid AFRAS PhD student in the research team "Modeling, Optimization and Dynamics of Structures in Civil Engineering" at the National school of applied sciences of Al Hoceima - Morocco, and holder of a civil engineering degree in 2016.

Abdelouafi EL GHOULBZOURI
Professor of civil engineering and research team director "Modeling, Optimization and Dynamics of Structures in Civil Engineering" at National school of applied sciences Al Hoceima-Morocco