



## THE EFFECT OF ADDED POINT MASSES ON THE GEOMETRICALLY NON-LINEAR VIBRATIONS OF SCSC RECTANGULAR PLATES

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### Abstract

A point mass added to a plate may have a significant effect on its linear and nonlinear dynamics, including frequencies, mode shapes and the forced response to external loading. In the present paper, a simply supported clamped simply supported clamped rectangular plate (SCSCRCP) carrying a point mass is examined. The expressions for the kinetic, linear and non-linear strain energies are derived by taking into account the effect of the added mass on the kinetic energy and the effect of the membrane forces induced by the non-linearity on the strain energy. The discretization of these expressions makes the mass tensor, the linear and non-linear rigidity tensors appear in a non-linear algebraic multimode amplitude equation, the iterative solution of which permit to obtain, in the neighborhood of the first non-linear mode, the basic SCSCRCP function amplitude dependent contribution coefficients. Nonlinear frequency response functions have been obtained for the first time, based on an iterative numerical solution in each case of the associated complete set of nonlinear algebraic equations. Such new results are useful for a better qualitative understanding allowing an optimal dynamic design of the rectangular plates with added masses.

Keywords: SCSC rectangular plates, added masses, non-linear free vibrations, non-linear forced vibrations

### 1. INTRODUCTION

The study of plate vibrations has been for a long time a subject of a great and continuous interest for researchers around the globe. Various geometries, boundary conditions and types of material properties are currently examined in order to guide designers and engineers working in structural dynamics. The effect of added masses on plate vibrations is of a crucial importance in many practical situations, because of the changes this induces in the plate natural frequencies, mode shapes and associated stress patterns, which may result in unexpected changes in the vibration manner of a SCSCRCP. The case of simply supported clamped simply supported clamped rectangular plates (SCSCRCP) carrying a point mass has been studied by [1] K. H. Low et al who examined a flat rectangular plate with mass components at random locations using the Rayleigh's energy approach, combined with experimental measurements and a finite element analysis. It was found that the fundamental frequency of the SCSCRCP decreases with increasing the mass ratio and increases when the mass is located away from the plate center. The comparison of the results with those obtained experimentally showed a relatively good agreement. [2] G. B. Chai investigated the same subject using the Rayleigh-Ritz method with multi-term trigonometric

functions. One-term solution was found good enough only if the point mass is placed at the plate center. [3] K. H. Low and his co-authors made comparisons of the frequencies of plates with added masses and various BC obtained theoretically by the Rayleigh-Ritz approach, and experimentally by a spectrum analysis and found that the analytical method can predict accurately the frequencies for one added mass. For more masses and higher modes, it was suggested to use more functions for better estimations. [4] Z. Beidouri et al investigated the geometrically non-linear transverse vibration of a rectangular plate, simply supported at three edges and clamped at one edge. The model based on Hamilton's principle and spectral analysis used in [5][6] was slightly modified to determine the frequency and mode shapes amplitude dependence for different plate aspect ratios. [7] Ding Zhou and Tianjian Ji studied a plate simply supported at two opposite edges and elastically supported along the other edges and attached to spring masses using an exact analytical solution leading to very accurate results. [8] Chai Gin Boay is studied free vibrations of rectangular plates with and without an added mass, and combinations of simply supported and clamped edges. To determine the modified natural frequencies, the Rayleigh-energy method with a single trigonometric function was used. Also, experimental results were given for SCSCRCP

carrying a point mass. It was found that the predicted first natural frequency of SCSCRCP carrying a centric mass was in a good agreement with the measurements, while that of a plate with an added eccentric mass was not well predicted. [9] A. J. Mcmillan and A. J. Keane presented a new approach to the vibration control over a wide frequency range and showed it to be possible to reach a bunch of vibration isolation with small added masses. [10] A. W. Leissa considered free vibrations of rectangular plates with a mixture of free, simply supported and clamped edge conditions. [11] J-S Wu et al studied free vibrations of rectangular plates with three types of added concentrated elements using different approaches and compared the results with those based on the finite element method. [12] R. G. Jacquot and W. Soedel dealt with the vibrations of structures carrying discrete dynamic systems at discrete points. [13] P. A. A. Laura et al studied the behavior of beams and plates carrying elastically a mass. [14] A. D. Kiureghian et al dealt with the dynamic behavior of a structure attached with a light equipment by perturbation methods. [15] Q.S. Li studied free vibrations of a rectangular plate with two opposite edges simply-supported carrying a line of concentrated masses or elastic line-supports using an exact approach. [16] H. Fakhreddine et al investigated the vibrations of non-linear forced vibration of a beam with added masses, they found that the stress increases with increasing the mass ratio and decreases close to the clamps. [17] D. Wang and M.I. Friswell analyzed the minimum support stiffness in order to rise the plate natural frequency and found the optimal attachment point and the related minimum stiffness. [18] P. Mahadevaswamy and B.S. Suresh dealt experimentally with the transverse vibrations of a clamped plate by vibratory flap excited harmonically and then compared his results with those based on a finite element analysis. [19] P.A. Martin and A. J. Hull studied the dynamic response of a thin plate carrying concentrated masses in linear regime and the results were compared to finite element computations. [20] X. Pang et al developed a new model for non-linear eigenvalue problem of the vibrations of a plate with elastically added masses. In this paper, non-linear vibrations of a SCSCRCP carrying an added point mass are investigated. First, comparisons of the natural frequencies of plates without and with an added mass are made. Then, the fundamental non-linear modes of various plates are given and compared to the linear ones, and new backbone curves indicating the effect of the added mass on the hardening type of non-linearity of SCSCRCP are displayed. Then, the non-linear frequency response function of a SCSCRCP carrying a point mass is presented for a wide frequency range.

## 2. NUMERICAL FORMULATION

Consider transverse vibrations of the SCSCRCP carrying a point mass shown in Fig. 1. The plate

membrane strain energy  $V_a$  induced by large vibration amplitudes can be written as in [5]:

$$V_a = \frac{3D}{2H^2} \int \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^2 dS \quad (1)$$

$D$  is the bending stiffness  $D = \frac{EH^3}{12(1-\nu^2)}$  and  $ds = dxdy$  is the elementary plate area,  $H$  is the plate thickness. The plate bending strain energy  $V_b$  is given by:

$$V_b = \frac{1}{2} \int D \left[ \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 - \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right] dS \quad (2)$$

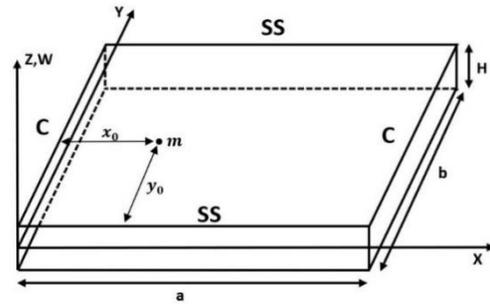


Fig. 1. The notation for a SCSCRCP with an added mass

The kinetic energy  $T$  of the plate and the added mass is:

$$T = \frac{1}{2} \rho H \int \left( \frac{\partial W}{\partial t} \right)^2 dxdy + \frac{1}{2} m \left( \frac{\partial W(x_0, y_0)}{\partial t} \right)^2 \quad (3)$$

If the space and time functions are supposed to be separable, the time supposed to be harmonic and the space term  $w(x, y)$  expanded in the form of a finite series, the transverse displacement function  $W$  can be written as:

$$W(x, y, t) = w(x, y) \sin(\omega t) = a_k w_k(x, y) \sin \omega t = a_{ij} w_{ij}(x, y) \sin(\omega t) \quad (4)$$

The functions  $w_{ij}(x, y)$  are obtained as products of  $x$  and  $y$  simply supported and clamped beam functions  $f_i(x)$  and  $g_j(y)$ :

$$w_{ij}(x, y) = f_i(x) g_j(y) \quad (5)$$

The kinetic, membrane and bending strain energy expressions become after discretization:

$$V_a = \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4(\omega t) \quad (6)$$

$$V_b = \frac{1}{2} a_i a_j k_{ij} \sin^2(\omega t) \quad (7)$$

$$T = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2(\omega t) \quad (8)$$

With  $b_{ijkl}$ ,  $k_{ij}$  and  $m_{ij}$  are the geometrical non-linearity, rigidity and mass tensors given by:

$$b_{ijkl} = \frac{3D}{H^2} \int \left( \frac{\partial w_i}{\partial x} \frac{\partial w_j}{\partial x} + \frac{\partial w_i}{\partial y} \frac{\partial w_j}{\partial y} \right) \left( \frac{\partial w_k}{\partial x} \frac{\partial w_l}{\partial x} + \frac{\partial w_k}{\partial y} \frac{\partial w_l}{\partial y} \right) dxdy$$

$$k_{ij} = \int D \left[ \left( \frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} \right) + 2(1-\nu) \left( \frac{\partial^2 w_i}{\partial x \partial y} \frac{\partial^2 w_j}{\partial x \partial y} - \frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_j}{\partial y^2} \right) \right] dxdy \quad (9)$$

$$m_{ij} = \rho H \int w_i w_j dx dy + m w_i(x_0, y_0) w_j(x_0, y_0) \quad (10)$$

Non-dimensional parameters are defined by:

$$w_i(x, y) = H w_i^* \left( \frac{x}{a}, \frac{y}{b} \right) = H w_i^*(x^*, y^*) \quad (11)$$

a and b being the plate length and width along the x and y axes. The previous tensors can be written as:

$$b^*_{ijkl} = 3 \int \left( \alpha^2 \frac{\partial w_i^*}{\partial x^*} \frac{\partial w_j^*}{\partial x^*} + \frac{\partial w_i^*}{\partial y^*} \frac{\partial w_j^*}{\partial y^*} \right) \left( \alpha^2 \frac{\partial w_k^*}{\partial x^*} \frac{\partial w_l^*}{\partial x^*} + \frac{\partial w_k^*}{\partial y^*} \frac{\partial w_l^*}{\partial y^*} \right) dx^* dy^* \quad (12)$$

$$k^*_{ij} = \int \left( \alpha^2 \frac{\partial^2 w_i^*}{\partial x^{*2}} + \frac{\partial^2 w_i^*}{\partial y^{*2}} \right) \left( \alpha^2 \frac{\partial^2 w_j^*}{\partial x^{*2}} + \frac{\partial^2 w_j^*}{\partial y^{*2}} \right) + 2(1 - \nu) \alpha^2 \left( \frac{\partial^2 w_i^*}{\partial x \partial y} \frac{\partial^2 w_j^*}{\partial x \partial y} - \frac{\partial^2 w_i^*}{\partial x^2} \frac{\partial^2 w_j^*}{\partial y^2} \right) dx^* dy^* \quad (13)$$

$$m^*_{ij} = \int w_i^* w_j^* dx^* dy^* + \eta w_i^*(x^*_0, y^*_0) w_j^*(x^*_0, y^*_0) \quad (14)$$

$\eta$  is the ratio of the added mass to the plate total mass  $\eta = \frac{m}{\rho H a b}$  and  $\alpha$  is the plate aspect ratio  $\alpha = \frac{b}{a}$ . The relationship between non-dimensional and dimensional tensors are:

$$b_{ijkl} = \frac{D a H^2}{b^3} b^*_{ijkl} \quad (15)$$

$$k_{ij} = \frac{D a H^2}{b^3} k^*_{ij} \quad (16)$$

$$m_{ij} = \rho H^3 a b m^*_{ij} \quad (17)$$

The equation of motion of the vibrating plate can be obtained by Hamilton's principle:

$$\delta \int_0^{2\pi/\omega} (V - T) dt = 0 \quad (18)$$

In which  $\delta$  indicates the variation of the integral, V and T are respectively the total strain and kinetic energies. This leads to the following set of n non-linear algebraic equations:

$$3a_i a_j a_k b^*_{ijk r} + 2a_i k^*_{ir} - 2\omega^{*2} a_i m^*_{ir} = 0 \quad r = 1, \dots, n \quad (19)$$

which may be written in a matrix form as:

$$3[B^*(A)]\{A\} + 2[K^*]\{A\} - 2\omega^{*2}[M^*]\{A\} = \{0\} \quad (20)$$

$\omega^{*2}$  can be expressed by pre-multiplying the last equation by  $\{A\}^T$ , leading to:

$$\omega^2 = \frac{a_i a_j k^*_{ij} + \frac{3}{2} a_i a_j a_k a_l b^*_{ijkl}}{a_i a_j m^*_{ij}} \quad (21)$$

With:

$$\omega^2 = \frac{D}{\rho h b^4} \omega^{*2} \quad (22)$$

Substituting equation (21) into the non-linear algebraic system leads to:

$$3a_i a_j a_k b^*_{ijk r} + 2a_i k^*_{ir} - 2 \frac{a_i a_j k^*_{ij} + \frac{3}{2} a_i a_j a_k a_l b^*_{ijkl}}{a_i a_j m^*_{ij}} a_i m^*_{ir} = 0 \quad r = 2, \dots, n \quad (23)$$

The results relative to non-linear vibrations have been obtained by solution of Equation (23) using the Harwell library NS01A routine and the complete non-linear tensor  $b^*_{ijk r}$  in order to get accurate results in the non-linear regime.

To study the response to a concentrated harmonic excitation force, a forcing term is added to the right-hand side of Equation (23), leading to:

$$\frac{3}{2} a_i a_j a_k b^*_{ijk r} + a_i k^*_{ir} - \omega^{*2} a_i m^*_{ir} = f_r^* \quad r = 1, \dots, n \quad (24)$$

$f_r^*$  is the dimensionless generalized force whose expression for a concentrated force F applied at the point of coordinates  $(x_0^*, y_0^*)$  is:

$$f_r^* = \frac{b^3 F}{a D H} w_r^*(x_0^*, y_0^*) \quad (25)$$

An iterative method using the Harwell library routine NS01A has been used to solve the n non-linear algebraic equations with n unknowns (24). It consisted on selecting a non-dimensional excitation frequency  $\omega^*$  and starting with an initial estimate for the n contributions  $(a_1, a_2, \dots, a_n)$ . The solution obtained was then taken as a new initial approximation for the following step for an excitation frequency  $\omega^* + \Delta\omega^*$ . The desired frequency segment was covered by repeating this process. It is worth noticing here that the routing sometimes diverges when passing through bifurcation points if the initial solution estimate is too far from the nearest solution.

### 3. NUMERICAL RESULTS AND DISCUSSION

The non-linear regime of a SCSCRCP with added masses is almost not discussed in the literature. Most of the papers deal only with linear frequencies and there is a wide obscure gap of information about the non-linear behavior of the SCSCRCP with an added mass including the non-linear frequencies and mode shapes, the hardening type of non-linearity accentuation and the non-linear forced response to harmonic excitation. This topic, addressed in the present work, is presented in what follows.

If only linear vibrations are considered, the term  $b^*_{ijk r}$  is omitted and Equation (19) reduces to:

$$a_i k^*_{ir} - \omega^{*2} a_i m^*_{ir} = 0 \quad r = 1, \dots, n \quad (26)$$

The classical eigen value problem (26) is solved using MATLAB Software to get the SCSCRCP with an added mass linear frequencies and mode shapes.

In the case of a SCSC square plate with no added masses, the non-dimensional parameter  $\eta$  equals 0 and the plate aspect ratio  $\alpha$  equals 1. The frequency obtained here using 9 by 9 symmetric and anti-symmetric functions in both directions have been compared to those given by Leissa [10], Whitney [21], Low et al [22], and G. B. Chai [23]. A good agreement is found between the present results and those of Leissa [10], Whitney [21] and G. B. Chai [23], as can be seen in Table 1. It is worth noticing that G. B. Chai [23] used 100 functions to generate his results, very close to those obtained here with much less computational effort.

Table 1. comparison of the natural frequencies of a square SCSCP with no mass added

mode	Leissa [10]	Whitney [21]	Low, Chai and Ng [22]	G. B. CHAI [23]	Present work
(1,1)	28.95	29.03	29.61	28.97	28.95
(1,2)	54.74	55.33	55.83	54.77	54.74
(2,1)	69.33	69.35	71.17	69.40	69.32
(2,2)	94.59	94.84	97.20	94.71	94.60

Table 1. Natural frequencies of a SCSCRCP with an added centric mass.  $\alpha = 0.59$ 

$\eta$	Experiment [22]	FEM [25]	R.Q. [8]	Chai Gin Boay [24]	Present work
0	62.76	67.62	68.48	67.11	67.08
0.13	51.52	53.91	51.40	48.75	50.08
0.28	41.94	44.45	42.35	38.83	39.85
0.41	34.14	38.90	37.20	33.48	34.60
0.56	31.6	34.73	33.30	26.59	30.51
0.69	28.51	31.88	30.62	25.99	27.93
0.82	26.85	29.61	28.49	24.96	25.90
0.98	24.86	27.50	26.49	23.09	23.93
1.10	24.02	26.12	25.19	21.88	22.71
1.23	23.41	24.82	23.96	20.76	21.57
1.38	22.24	23.62	22.82	19.72	20.46
1.53	22.00	22.54	21.79	18.79	19.48
1.67	21.29	21.66	20.95	18.03	18.71
1.81	20.44	20.87	20.20	17.36	18.02
1.94	20.11	20.17	19.53	16.76	17.44
2.08	19.47	19.53	18.92	16.22	16.87
2.22	18.69	18.95	18.36	15.73	16.35
2.36	18.17	18.42	17.85	15.28	15.88
2.50	17.88	17.93	17.38	14.86	15.45
2.64	17.44	17.47	16.94	14.48	15.05
2.78	17.23	17.05	16.54	14.12	14.68

For comparison purposes for a plate with an added centric mass, the SCSCP studied in [24] has been examined by the present method using, due to the symmetry, only 4 symmetric functions in both directions, which led to 16 plate functions in total.

Table 2 presents the frequencies obtained experimentally in [22] and those calculated here for a SCSCRCP ( $\alpha = 0.59$ ) with an added centric mass. The difference between experimental and semi-analytical results does not exceed 4% for added mass ratios  $\eta$  less than unity. For higher values, up to  $\eta = 2.78$ , the difference increases to 15%. This may be explained by the fact that the added masses are attached together vertically, leading to an increase in the moment of inertia when multiple masses are used, which is not taken into consideration here.

The presence of the added mass concentrates the vibration at the mass location. It is also noticed that for eccentric added masses, the point of maximum deflection is moved from the plate center to the mass location and that phenomenon is clearer for small plate aspect ratios and high mass ratios.

Concerning the fundamental mode shape, it is affected by the simple supports, especially in the non-linear regime. Fig. 3 presents the normalized sections of the first non-linear mode shape corresponding to  $y^*=0.5$  for SCSCRCP ( $\alpha=0.6$ ) carrying a point mass ( $\eta=0.2$ ) at its center. Fig. 4 presents the same conditions but along the line  $x^*=0.5$ . It is clear that the changes appear along the  $y^*$  direction which is close to the simple supports and that the added mass tends to make the shape of the mode around the mass location straighter with an increase in the vibration amplitude. The clamped edges limit the changes at large vibration amplitudes on the contrary to the simply supported edges.

Fig. 5 presents the normalized section of the plate first non-linear mode shape corresponding to  $y^*=0.5$  and  $\alpha = 0.6$  and an added mass ( $\eta = 0.2$ ) placed at the point of coordinates (0.25,0.5). The normalized sections correspond to different amplitudes as indicated in the figure. It appears that the maximum displacement  $W_{max}$  moves towards the mass location with increasing the vibration amplitude and that the curvature increases in the largest area between the maximum of the mode and the clamps.

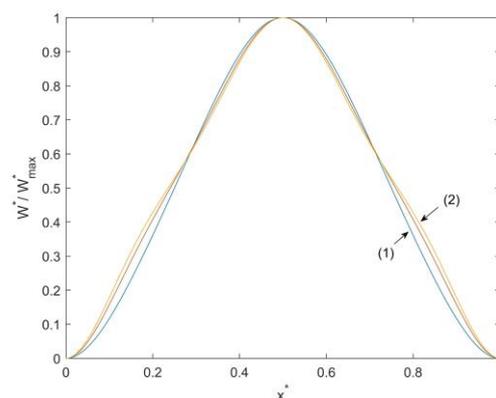


Fig. 2. Normalized sections of the first non-linear mode shape corresponding to  $y^* = 0.5$  and  $\alpha = 0.6$  carrying a concentrated mass at the plate center  $\eta = 0.2$ . (1) lowest amplitude, (2): highest amplitude

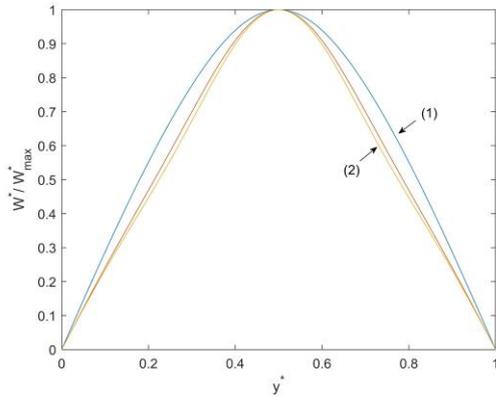


Fig. 3. Normalized sections of the first non-linear mode shape corresponding to  $x^* = 0.5$  and  $\alpha = 0.6$  for a plate carrying a concentrated mass at its center  $\eta = 0.2$ . (1) lowest amplitude, (2) highest amplitude

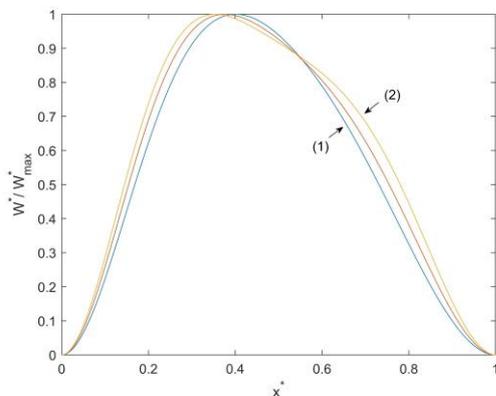


Fig. 4. Normalized sections of the first non-linear mode shape corresponding to  $y^* = 0.5$  and  $\alpha = 0.6$  for a plate carrying concentrated mass at  $(0.25,0.5)$   $\eta = 0.2$ . (1) lowest amplitude, (2) highest amplitude

The non-linear behavior of a SCSCRCP carrying a mass is examined in this paper in order to investigate the ways the added mass hardens or softens its non-linearity type. This appears in the joint figures displaying the maximum amplitude  $W_{max}$  versus the non-linear frequency parameter. Fig. 6 presents the backbone curves of a SCSCRCP ( $\alpha = 0.6$ ) with no mass added and with an added centric mass for mass ratios  $\eta$  equal to 0.1 and 0.2. It is observed that the added centric mass decreases the hardening type of non-linearity, which decreases also with increasing the added mass ratio.

The backbone curves presented in Fig. 7 for SCSCRCP without and with an added mass located at  $(0.25,0.5)$ , for mass ratios  $\eta = 0.1, 0.2$ , reveal that for  $\eta = 0.1$ , the non-linearity is not affected. For  $\eta = 0.2$ , the hardening effect is more reduced if the mass location is closer to the clamped edge than to the simply supported edge. Fig. 8 shows the backbone curves of SCSCRCP with no added mass and with an added centric mass located at  $(0.5, 0.25)$  for  $\eta = 0.1, 0.2$ . It is clear that the added mass at that location

increases the hardening type of non-linearity, which may be due to the fact that in this case the mass is closer to the simply supported than to the clamped edge.

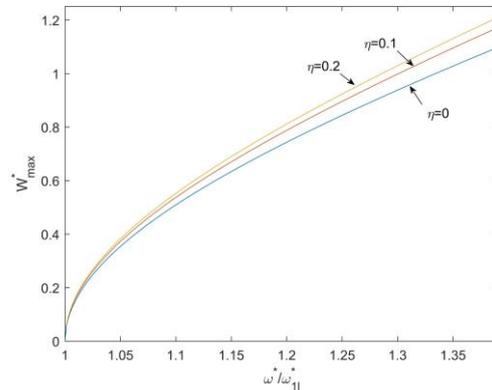


Fig. 5. the backbone curve of a SCSCR plate with an aspect ratio  $\alpha = 0.6$  and no mass added and with an added mass located at the plate center

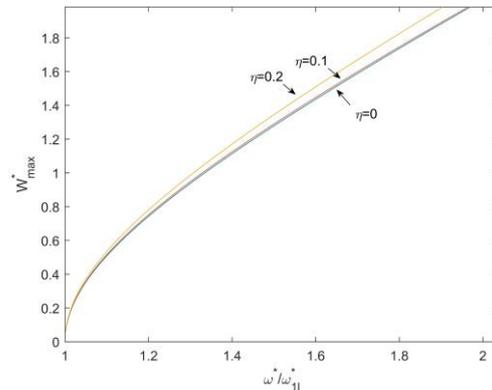


Fig. 6. the backbone curve of a SCSCR plate with an aspect ratio  $\alpha=0.6$  and no mass added and with an added mass located at  $(0.25,0.5)$

The SCSCRCP response to a harmonic concentrated force applied at different locations is investigated in this paper and plots of the obtained non-linear frequency response functions (NLFRF) are given. The mechanical characteristics adopted here are:  $a = 0,25m$ ,  $b = 0,15m$ ,  $h = 0,0005m$ ,  $\rho = 7850Kg/m^3$ ,  $E = 198.10^9Pa$  and  $\nu = 0.3$ . Fig. 9 presents the NLFRF a SCSCRCP with no added mass and with an added mass located at its center, subjected to a harmonic force applied also at the plate center. The SCSCRCP exhibits a hardening type of non-linearity and the added mass decreases the non-linearity as can be seen in the vicinity of the first, fourth and eighth modes. The second and third modes are not excited by the centric harmonic force. Fig. 9 presents the NLFRF of a SCSCRCP ( $\alpha = 0.6$ ) with no added mass and with an added mass located at  $(0.25,0.5)$  under a harmonic centric excitation. It is worth noticing that the mass located at  $(0.25,0.5)$  activates the response to a centric excitation of the second mode but not that of the third mode. This is due to the fact the added mass moves the nodal line of the second mode from the

plate middle line. Also, it activates the response to the sixth mode. This observation may be useful in understanding that an added centric or eccentric mass may affect unexpectedly the dynamics of a SCSCR plate which requires a careful analysis.

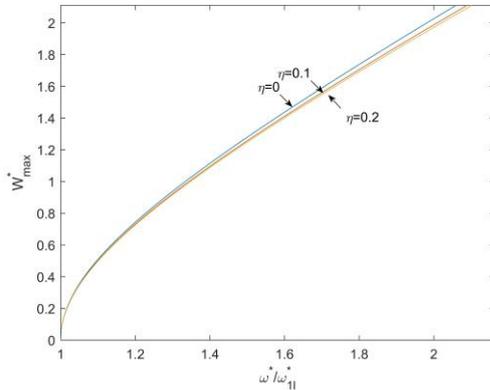


Fig. 7. the backbone curve of a SCSCR plate with an aspect ratio  $\alpha=0.6$  and no mass added and with an added mass located at  $(0.5,0.25)$

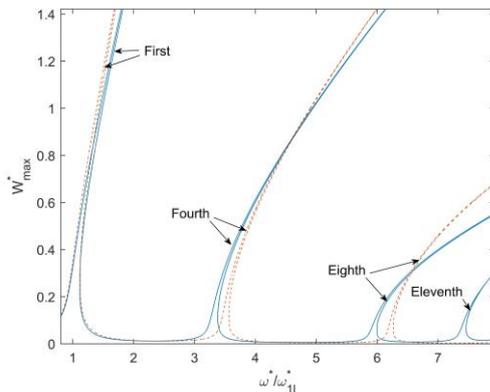


Fig. 8. the non-linear frequency response function of a plate  $\alpha = 0.6$  subjected to a centric harmonic concentrated force  $F = 0.3N$ . The continuous line: no added mass. The discrete line: an added mass located at the plate center  $\eta = 0.1$

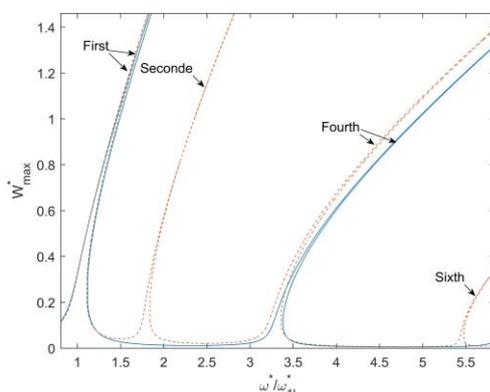


Fig. 9. the non-linear frequency response function of a plate  $\alpha = 0.6$  subjected to a centric harmonic concentrated force  $F = 0.3N$ . The continuous line: no added mass. The discrete line: an added mass located at  $(0.25,0.5)$   $\eta = 0.1$

#### 4. CONCLUSION

The semi analytical model presented here for linear and non-linear vibrations of a SCSCR plate with an added point mass has been first successfully validated by comparing the results obtained for the linear frequencies with those available in the literature. The presence of the added mass changes the mode shape by concentrating the deformation at the added mass location, and this effect appears more for low plate aspect ratios. In the case of non-linear vibrations, the added mass modifies the configuration of the mode shape as in linear vibrations with slight deformations beside the location of the added mass in the simply supported direction. Also, it is worth noticing that the hardening non-linearity type of SCSCR plates increases if the added point mass is placed in the simple supports direction and decreases if the mass is placed at the plate center or in the clamped supports direction. Also, the presence of an added eccentric mass activates the participation in the response of antisymmetric modes to a centric harmonic excitation because of the changes it induces in the whole modal distributions, including frequencies and nodal lines. This shows that a mass addition may lead, if not well controlled, to unexpected situations, requiring careful attention. Also, non-linear frequency response functions have been obtained for the first time, based on iterative numerical solution in each case of the complete set of non-linear algebraic equations, for various SCSCR carrying a centric or an eccentric added mass. The obtained results by the present model allows predicting the vibration characteristics of a plate with an added mass. The necessary precautions could then be made when diagnosing a vibrating structure since the presence of the added mass changes the configuration of the linear and non-linear mode shapes and the response to a forced excitation and the changes depend on the position of the added mass.

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#### REFERENCES

1. Low KH, Chai GB, Lim TM, Sue SC. Comparisons of experimental and theoretical frequencies for rectangular plates with various boundary conditions and added masses. *International Journal of Mechanical Sciences*. 1998;40(11), 1119-1131.

2. Boay CG. Free vibration of rectangular isotropic plates with and without a concentrated mass. *Computers and structures*. 1993;48(3):529-533.
3. Low KH, Chai GB. An improved model for predicting fundamental frequencies of plates carrying multiple masses. 1997.
4. Beidouri Z, Benamar R, El Kadiri M. Geometrically non-linear transverse vibrations of C-S-S-S and C-S-C-S rectangular plates. *International Journal of Non-Linear Mechanics*. 2006;41(1):57-77.
5. Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, Part II: fully clamped rectangular isotropic plates. *Journal of Sound and Vibration*. 1993;164(2):295-316.
6. El Kadiri M, Benamar R. Improvement of the semi-analytical method, for determining the geometrically non-linear response of thin straight structures: part II-first and second non-linear mode shapes of fully clamped rectangular plates. *Journal of Sound and Vibration*. 2002;257(1):19-62.
7. Zhou D, Ji T. Free vibration of rectangular plates with attached discrete sprung masses. *Shock and Vibration*. 2012;19(1):101-112. <https://doi.org/10.3233/SAV-2012-0618>.
8. Boay CG. Free vibration of rectangular isotropic plates with and without a concentrated mass. *Computers and Structures*. 1993;48(3):529-533.
9. McMillan AJ, Keane AJ. Shifting resonances from a frequency band by applying concentrated masses to a thin rectangular plate. *Journal of Sound and Vibration*. 1996;192(2): 549-652.
10. Leissa AW. *Vibration of shells*. Scientific and Technical Information Office, National Aeronautics and Space Administration. 1973;288.
11. Wu JS, Chou HM, Chen DW. Free vibration analysis of a rectangular plate carrying multiple various concentrated elements. *Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body Dynamics*. 2003;217(2):171-183. <https://doi.org/10.1243/146441903321898656>.
12. Jacquot RG, Soedel W. Vibrations of elastic surface systems carrying dynamic elements. *The Journal of the Acoustical Society of America*. 1970;47(5B):1354-1358.
13. Laura PAA, Filipich CP, Cortinez VH. Vibrations of beams and plates carrying concentrated masses. *Journal of Sound Vibration*. 1987;117(3): 459-465.
14. Der Kiureghian, A, Sackman JL, Nour Omid B. *Dynamic response of light equipment in structures*. Earthquake Engineering Center, College of Engineering, University of California. 1981.
15. Li QS. An exact approach for free vibration analysis of rectangular plates with line concentrated mass and elastic line-support. *International Journal of Mechanical Sciences*. 2003;45(4):669-685. [https://doi.org/10.1016/S0020-7403\(03\)00110-3](https://doi.org/10.1016/S0020-7403(03)00110-3).
16. Fakhreddine H, Adri A, Chajdi M, Rifai S, Benamar R. A multimode approach to geometrically non-linear forced vibration of beams carrying point masses. *Diagnostyka*. 2020;21(4):23-33. <https://doi.org/10.29354/diag/128603>.
17. Wang D, Friswell MI. Support position optimization with minimum stiffness for plate structures including support mass. *Journal of Sound and Vibration*. 2021;499:116003. <https://doi.org/10.1016/j.jsv.2021.116003>.
18. Mahadevaswamy P. Et Suresh BS. Optimal mass ratio of vibratory flap for vibration control of clamped rectangular plate. *Ain Shams Engineering Journal*. 2016;7(1):335-345. <https://doi.org/10.1016/j.asej.2015.11.014>.
19. Martin PA, Et Hull, Andrew J. Dynamic response of an infinite thin plate loaded with concentrated masses. *Wave Motion*. 2020;98:102643. <https://doi.org/10.1016/j.wavemoti.2020.102643>.
20. Pang X, Sun J, Zhang Z. FE-holomorphic operator function method for nonlinear plate vibrations with elastically added masses. *Journal of Computational and Applied Mathematics*. 2022:114156.
21. Whitney JM. *Structural Analysis of Laminated Anisotropic Plates* (Lancaster, PA: Technomic). 1987.
22. Low KH, Chai GB, Ng CK. Experimental and analytical study of the frequencies of an SCSC plate carrying a concentrated mass. 1993;115(4):391-396. <https://doi.org/10.1115/1.2930362>.
23. Chai GB. Frequency analysis of a SCSC plate carrying a concentrated mass. *Journal of sound and vibration*. 1995;179(1):170-177.
24. Boay CG. Frequency analysis of rectangular isotropic plates carrying a concentrated mass. *Computers and structures*. 1995;56(1):39-48.
25. Low KH, Ng CK, Ong YK. Comparative study of frequencies for plates carrying mass. *Journal of engineering mechanics*. 1993;119(5):917-937.
26. Wądołowski M, Pankiewicz J, Markuszewski D. Application for analysis of the multiple coherence function in diagnostic signal separation processes. *Vibrations in Physical Systems*. 2020;31(3):2020324. <https://doi.org/10.21008/j.0860-6897.2020.3.24>.

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